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REALE ACCADEMIA DELLE SCIENZE DI TORINO

Plana—II-68

THE DOCTRINE OF CHANCES:

OR,

A METHOD of Calculating the Probabilities of Events in PLAY.

THE THIRD EDITION,
Fuller, Clearer, and more Correct than the Former.

By A. DE MOIVRE,
Fellow of the ROYAL SOCIETY, *and Member of the* ROYAL ACADEMIES OF SCIENCES *of* Berlin *and* Paris.

ACCADEMIA
R. DELLE SCIENZE
DI TORINO

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To the RIGHT HONOURABLE the
Lord *CARPENTER*.*

MY LORD,

THERE are many People in the World who are prepoſſeſſed with an Opinion, that the Doctrine of Chances has a Tendency to promote Play; but they ſoon will be undeceived, if they think fit to look into the general Deſign of this Book: in the mean time it will not be improper to inform them, that your Lordſhip is pleaſed to eſpouſe the Patronage of this ſecond Edition; which your ſtrict Probity, and the diſtinguiſhed Character you bear in the World, would not have permitted, were not their Apprehenſions altogether groundleſs.

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* This Dedication was prefixed to the 2d Edition.

DEDICATION.

Your Lordſhip does eaſily perceive, that this Doctrine is ſo far from encouraging Play, that it is rather a Guard againſt it, by ſetting in a clear Light, the Advantages and Diſadvantages of thoſe Games wherein Chance is concerned.

Beſides the Endowments of the Mind which you have in common with Thoſe whoſe natural Talents have been cultivated by the beſt Education, you have this particular Happineſs, that you underſtand, in an eminent Degree, the Principles of Political Arithmetic, the Nature of our Funds, the National Credit, and its Influence on public Affairs.

As one Branch of this uſeful Knowledge extends to the Valuation of Annuities founded on the Contingencies of Life, and that I have made it my particular Care to facilitate and improve the Rules I have formerly given on that Subject; I flatter myſelf with a favourable Acceptance of what is now, with the greateſt Deference, ſubmitted to your Judgment, by,

My Lord,

Your Lordſhip's

Moſt Obedient and

Moſt Obliged,

Humble Servant,

A. de Moivre.

PREFACE.[*]

'TIS now about Seven Years, ſince I gave a Specimen in the Philoſophical Tranſactions, *of what I now more largely treat of in this Book. The occaſion of my then undertaking this Subject was chiefly owing to the Deſire and Encouragement of the Honourable* [†] Francis Robartes *Eſq; who, upon occaſion of a French Tract, called,* L'Analyſe des Jeux de Hazard, *which had lately been publiſhed, was pleaſed to propoſe to me ſome Problems of much greater difficulty than any he had found in that Book; which having ſolved to his Satisfaction, he engaged me to methodize thoſe Problems, and to lay down the Rules which had led me to their Solution. After I had proceeded thus far, it was enjoined me by the Royal Society, to communicate to them what I had diſcovered on this Subject: and thereupon it was ordered to be publiſhed in the Tranſactions, not ſo much as a matter relating to Play, but as containing ſome general Speculations not unworthy to be conſidered by the Lovers of Truth.*

I had not at that time read any thing concerning this Subject, but Mr. Huygen's *Book* de Ratiociniis in Ludo Aleæ, *and a little Engliſh Piece (which was properly a Tranſlation of it) done by a very ingenious Gentleman, who, tho' capable of carrying the matter a great deal farther, was contented to follow his Original; adding only to it the computation of the Advantage of the Setter in the Play called* Hazard, *and ſome few things more. As for the French Book, I had run it over but curſorily, by reaſon I had obſerved that the Author chiefly inſiſted on*

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[*] *This Preface was written in* 1717.

[†] *Now Earl of* RADNOR.

the Method of Huygens, *which I was abſolutely reſolved to reject, as not ſeeming to me to be the genuine and natural way of coming at the Solution of Problems of this kind.*

I had ſaid in my Specimen, that Mr. Huygens *was the firſt who had publiſhed the Rules of this Calculation, intending thereby to do juſtice to a Man who had well deſerved of the Public; but what I then ſaid was miſinterpreted, as if I had deſigned to wrong ſome Perſons who had conſidered this matter before him: and a Paſſage was cited againſt me out of* Huygen's *Preface, in which he ſaith,* Sciendum vero quod jam pridem, inter Præſtantiſſimos totâ Galliâ Geometras, Calculus hic fuerit agitatus; ne quis indebitam mihi primæ Inventionis gloriam hac in re tribuat. *But what follows immediately after, had it been minded, might have cleared me from any Suſpicion of injuſtice. The words are theſe,* Cæterum illi difficillimis quibuſque Quæſtionibus ſe invicem exercere ſoliti, methodum ſuam quiſque occultam retinuere, adeo ut a primis elementis hanc materiam evolvere mihi neceſſe fuerit. *By which it appears, that tho'* Mr. Huygens *was not the firſt who had applied himſelf to thoſe ſorts of Queſtions, he was nevertheleſs the firſt who had publiſhed Rules for their Solution; which is all that I affirmed.*

Such a Tract as this is may be uſeful to ſeveral ends; the firſt of which is, that there being in the World ſeveral inquiſitive Perſons, who are deſirous to know what foundation they go upon, when they engage in Play, whether from a motive of Gain, or barely Diverſion, they may, by the help of this or the like Tract, gratify their curioſity, either by taking the pains to underſtand what is here Demonſtrated, or elſe making uſe of the Concluſions, and taking it for granted that the Demonſtrations are right.

Another uſe to be made of this Doctrine of Chances is, that it may ſerve in Conjunction with the other parts of the Mathematicks, as a fit Introduction to the Art of Reaſoning; it being known by experience that nothing can contribute more to the attaining of that Art, than the conſideration of a long Train of Conſequences, rightly deduced from undoubted Principles, of which this Book affords many Examples. To this may be added, that ſome of the Problems about Chance having a great appearance of Simplicity, the Mind is eaſily drawn into a belief, that their Solution may be attained by the meer Strength of natural good Senſe; which generally proving otherwiſe, and the Miſtakes occaſioned thereby being not unfrequent, 'tis preſumed that a Book of this Kind, which teaches to diſtinguiſh Truth from what ſeems ſo nearly to reſemble it, will be looked upon as a help to good Reaſoning.

Among the ſeveral Miſtakes that are committed about Chance, one of the moſt common and leaſt ſuſpected, is that which relates to Lotteries. *Thus, ſuppoſing a Lottery wherein the proportion of the Blanks to the Prizes is as five to one; 'tis very natural to conclude, that therefore five Tickets are requiſite for the Chance of a Prize; and yet it may be proved, Demonſtratively, that four Tickets are more than ſufficient for that purpoſe, which will be confirmed by often repeated Experience. In the like manner, ſuppoſing a Lottery wherein the proportion of the Blanks to the Prizes is as Thirty-nine to One, (ſuch as was the Lottery of* 1710) *it may be proved, that in twenty eight Tickets, a Prize is as likely to be taken as not; which tho' it may ſeem to contradict the common Notions, is nevertheleſs grounded upon infallible Demonſtration.*

When the Play of the Royal Oak was in uſe, ſome Perſons who loſt conſiderably by it, had their Loſſes chiefly occaſioned by an Argument of which they could not perceive the Fallacy. The Odds againſt any particular Point of the Ball were One and Thirty to One, which intitled the Adventurers, in caſe they were winners, to have thirty two Stakes returned, including their own; inſtead of which they having but Eight and Twenty, it was very plain that on the ſingle account of the diſadvantage of the Play, they loſt one eighth part of all the Money they played for. But the Maſter of the Ball maintained that they had no reaſon to complain; ſince he would undertake that any particular point of the Ball ſhould come up in Two and Twenty Throws; of this he would offer to lay a Wager, and actually laid it when required. The ſeeming contradiction between the Odds of One and Thirty to One, and Twenty-two Throws for any Chance to come up, ſo perplexed the Adventurers, that they begun to think the Advantage was on their ſide; for which reaſon they played on and continued to loſe.

The Doctrine *of* Chances *may likewiſe be a help to cure a Kind of Superſtition, which has been of long ſtanding in the World, viz. that there is in Play ſuch a thing as* Luck, *good or bad. I own there are a great many judicious people, who without any other Aſſiſtance than that of their own reaſon, are ſatisfied, that the Notion of* Luck *is meerly Chimerical; yet I conceive that the ground they have to look upon it as ſuch, may ſtill be farther inforced from ſome of the following Conſiderations.*

If by ſaying that a Man has had good Luck, nothing more was meant than that he has been generally a Gainer at play, the Expreſſion might be allowed as very proper in a ſhort way of ſpeaking: But if the Word Good Luck *be underſtood to ſignify a certain predominant quality, ſo inherent in a Man, that he muſt win whenever he Plays, or at leaſt win oftner than loſe, it may be denied that there is any ſuch thing in nature.*

The Aſſerters of Luck *are very ſure from their own Experience, that at ſome times they have been very Lucky, and that at other times they have had a prodigious Run of* ill Luck *againſt them, which whilſt it continued obliged them to be very cautious in engaging with the Fortunate; but how Chance ſhould produce thoſe extraordinary Events, is what they cannot conceive: They would be glad, for Inſtance, to be Satisfied, how they could loſe Fifteen Games together at* Piquet, *if* ill Luck *had not ſtrangely prevailed againſt them. But if they will be pleaſed to conſider the Rules delivered in this Book, they will ſee, that though the Odds againſt their loſing ſo many times together be very great,* viz. 32767 *to* 1, *yet that the Poſſibility of it is not deſtroyed by the greatneſs of the* Odds, *there being One Chance in* 32768 *that it may ſo happen; from whence it follows, that it was ſtill poſſible to come to paſs without the Intervention of what they call* Ill Luck.

Beſides, This Accident of loſing Fifteen times together at Piquet, *is no more to be imputed to ill Luck, than the Winning with one ſingle Ticket the higheſt Prize, in a Lottery of* 32768 *Tickets, is to be imputed to good Luck, ſince the Chances in both Caſes are perfectly equal. But if it be ſaid that Luck has been concerned in this latter Caſe, the Anſwer will be eaſy; for let us ſuppoſe Luck not exiſting, or at leaſt let us ſuppoſe its Influence to be ſuſpended, yet the higheſt Prize muſt fall into ſome Hand or other, not by Luck, (for by the Hypotheſis that has been laid aſide) but from the meer neceſſity of its falling ſomewhere.*

Thoſe who contend for Luck, may, if they pleaſe, alledge other Caſes at Play, much more unlikely to happen than the Winning or Loſing fifteen Games together, yet ſtill their Opinion will never receive any Addition of Strength from ſuch Suppoſitions: For, by the Rules of Chance, a time may be computed, in which thoſe Caſes may as probably happen as not; nay, not only ſo, but a time may be computed in which there may be any proportion of Odds for their ſo happening.

But ſuppoſing that Gain and Loſs were ſo fluctuating, as always to be diſtributed equally, whereby Luck would certainly be annihilated; would it be reaſonable in this Caſe to attribute the Events of Play to Chance alone? I think, on the contrary, it would be quite otherwiſe, for then there would be more reaſon to ſuſpect that ſome unaccountable Fatality did rule in it: Thus, if two Perſons play at Croſs *and* Pile, *and Chance alone be ſuppoſed to be concerned in regulating the fall of the Piece, is it probable that there ſhould be an Equality of Heads and Croſſes? It is Five to Three that in four times there will be an inequality; 'tis Eleven to Five in ſix,* 93 *to* 35 *in Eight, and about* 12 *to* 1 *in a hundred times: Wherefore Chance alone by its Nature conſtitutes the Inequalities of Play, and there is no need to have recourſe to Luck to explain them.*

Further, the ſame Arguments which explode the Notion of Luck, may, on the other ſide, be uſeful in ſome Caſes to eſtabliſh a due compariſon between Chance and Deſign: We may imagine Chance and Deſign to be, as it were, in Competition with each other, for the production of ſome ſorts of Events, and may calculate what Probability there is, that thoſe Events ſhould be rather owing to one than to the other. To give a familiar Inſtance of this, Let us ſuppoſe that two Packs of Piquet-Cards being ſent for, it ſhould be perceived that there is, from Top to Bottom, the ſame Diſpoſition of the Cards in both Packs; let us likewiſe ſuppoſe that, ſome doubt ariſing about this Diſpoſition of the Cards, it ſhould be queſtioned whether it ought to be attributed to Chance, or to the Maker's Deſign: In this Caſe the Doctrine of Combinations decides the Queſtion; ſince it may be proved by its Rules, that there are the Odds of above 263130830000 *Millions of Millions of Millions of Millions to One, that the Cards were deſignedly ſet in the Order in which they were found.*

From this laſt Conſideration we may learn, in may Caſes, how to diſtinguiſh the Events which are the effect of Chance, from thoſe which are produced by Deſign: The very Doctrine that finds Chance where it really is, being able to prove by a gradual Increaſe of Probability, till it arrive at Demonſtration, that where Uniformity, Order and Conſtancy reſide, there alſo reſide Choice and Deſign.

Laſtly, *One of the principal Uſes to which this* Doctrine of Chances *may be applied, is the diſcovering of ſome Truths, which cannot fail of pleaſing the Mind, by their Generality and Simplicity; the admirable Connexion of its Conſequences will increaſe the Pleaſure of the Diſcovery; and the ſeeming Paradoxes wherewith it abounds, will afford very great matter of Surprize and Entertainment to the Inquiſitive. A very remarkable Inſtance of this nature may be ſeen in the prodigious Advantage which the repetition of Odds will amount to; Thus, Suppoſing I play with an Adverſary who allows me the Odds of* 43 *to* 40, *and agrees with me to play till* 100 *Stakes are won or loſt on either ſide, on condition that I give him an Equivalent for the Gain I am intitled to by the Advantage of my Odds; the Queſtion is, what I am to give him, on ſuppoſing we play a Guinea a Stake: The Anſwer is* 99 *Guineas and above* 18 *Shillings*, which will ſeem almoſt incredible, conſidering the ſmallneſs of the Odds of* 43 *to* 40. *Now let the Odds be in any Proportion given, and let the Number of Stakes be played for be never ſo great, yet one general Concluſion will include all the poſſible Caſes, and the application of it to Numbers may be wrought in leſs than a Minute's time.*

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* *Guineas were then at* 21$^{ſh.}$ 6$^{d.}$

I have explained, in my Introduction *to the following Treatiſe, the chief Rules on which the whole Art of Chances depends; I have done it in the plaineſt manner that I could think of, to the end it might be (as much as poſſible) of general Uſe. I flatter my ſelf that thoſe who are acquainted with Arithmetical Operations, will, by the help of the* Introduction *alone, be able to ſolve a great Variety of Queſtions depending on Chance: I wiſh, for the ſake of ſome Gentlemen who have been pleaſed to ſubſcribe to the printing of my Book, that I could every where have been as plain as in the* Introduction; *but this was hardly practicable, the Invention of the greateſt part of the Rules being intirely owing to* Algebra; *yet I have, as much as poſſible, endeavoured to deduce from the Algebraical Calculation ſeveral practical Rules, the Truth of which may be depended upon, and which may be very uſeful to thoſe who have contented themſelves to learn only common Arithmetick.*

On this occaſion, I muſt take notice to ſuch of my Readers as are well verſed in Vulgar Arithmetick, that it would not be difficult for them to make themſelves Maſters, not only of all the practical Rules in this Book, but alſo of more uſeful Diſcoveries, if they would take the ſmall Pains of being acquainted with the bare Notation of Algebra, *which might be done in the hundredth part of the Time that is ſpent in learning to write* Short-hand.

One of the principal Methods I have made uſe of in the following Treatiſe, has been the Doctrine of Combinations, *taken in a Senſe ſomewhat more extenſive, than as it is commonly underſtood. The Notion of Combinations being ſo well fitted to the Calculation of Chance, that it naturally enters the Mind whenever an Attempt is made towards the Solution of any Problem of that kind. It was this that led me in courſe to the Conſideration of the Degrees of Skill in the Adventurers at Play, and I have made uſe of it in moſt parts of this Book, as one of the* Data *that enter the Queſtion; it being ſo far from perplexing the Calculation, that on the contrary it is rather a Help and an Ornament to it: It is true, that this Degree of Skill is not to be known any other way than from Obſervation; but if the ſame Obſervation conſtantly recur, 'tis ſtrongly to be preſumed that a near Eſtimation of it may be made: However, to make the Calculation more preciſe, and to avoid cauſing any needleſs Scruples to thoſe who love Geometrical Exactneſs, it will be eaſy, in the room of the word* Skill, *to ſubſtitute a Greater or Leſs Proportion of* Chances *among the Adventurers, ſo as each of them may be ſaid to have a certain Number of Chances to win one ſingle Game.*

The general Theorem invented by Sir Iſaac Newton, *for raiſing a Binomial to any Power given, facilitates infinitely the Method of* Combinations, *repreſenting in one View the Combination of all the Chances, that can happen in any given Number of Times. 'Tis by the help of that Theorem, joined with ſome other Methods, that I have been able to find practical Rules for the ſolving a great Variety of difficult Queſtions, and to reduce the Difficulty to a ſingle Arithmetical Multiplication, whereof ſeveral Inſtances may be ſeen in the 46th Page of this Book.*

Another Method I have made uſe of, is that of Infinite Series, *which in many caſes will ſolve the Problems of Chance more naturally than Combinations. To give the Reader a Notion of this, we may ſuppoſe two Men at Play throwing a Die, each in their Turns, and that he is to be reputed the Winner who ſhall firſt throw an Ace: It is plain, that the Solution of this Problem cannot ſo properly be reduced to Combinations, which ſerve chiefly to determine the proportion of Chances between the Gameſters, without any regard to the Priority of Play. 'Tis convenient therefore to have recourſe to ſome other Method, ſuch as the following: Let us ſuppoſe that the firſt Man, being willing to compound with his Adverſary for the Advantage he is intitled to from his firſt Throw, ſhould aſk him what Conſideration he would allow to yield it to him; it may naturally be ſuppoſed that the Anſwer would be one Sixth part of the Stake, there being but Five to One againſt him, and that this Allowance would be thought a juſt Equivalent for yielding his Throw. Let us likewiſe ſuppoſe the ſecond Man to require in his Turn to have one ſixth part of the remaining Stake for the Conſideration of his Throw; which being granted, and the firſt Man's Right returning in courſe, he may claim again one ſixth part of the Remainder, and ſo on alternately, till the whole Stake be exhauſted: But this not being to be done till after an infinite number of Shares be thus taken on both Sides, it belongs to the Method of* Infinite Series *to aſſign to each Man what proportion of the Stake he ought to take at firſt, ſo as to anſwer exactly that fictitious Diviſion of the Stake* in infinitum; *by means of which it will be found, that the Stake ought to be divided between the contending Parties into two parts, reſpectively proportional to the two Numbers 6 and 5. By the like Method it would be found that if there were Three or more Adventurers playing on the conditions above deſcribed, each Man, according to the Situation he is in with reſpect to Priority of Play, might take as his due ſuch part of the Stake, as is expreſſible by the correſponding Term of the proportion of 6 to 5, continued to ſo many Terms as there are Gameſters; which in the caſe of Three Gameſters, for Inſtance, would be the Numbers 6, 5, and* $4\frac{1}{6}$, *or their Proportionals 36, 30, and 25.*

Another Advantage of the Method of Infinite Series is, that every Term of the Series includes ſome particular Circumſtance wherein the

Gameſters may be found, which the other Methods do not; and that a few of its Steps are ſufficient to diſcover the Law of its Proceſs. The only Difficulty which attends this Method, being that of ſumming up ſo many of its Terms as are requiſite for the Solution of the Problem propoſed: But it will be found by Experience, that in the Series reſulting from the Conſideration of moſt Caſes relating to Chance, the Terms of it will either conſtitute a Geometric Progreſſion, which by the known Methods is eaſily ſummable; or elſe ſome other ſort of Progreſſion, whoſe nature conſiſts in this, that every Term of it has to a determinate number of the preceding Terms, each being taken in order, ſome conſtant relation; in which caſe I have contrived ſome eaſy Theorems, not only for finding the Law of that Relation, but alſo for finding the Sums required; as may be ſeen in ſeveral places of this Book, but particularly from page 220 *to page* 230.

A Third Advantage of the Method of Infinite Series is, that the Solutions derived from it have a certain Generality and Elegance, which ſcarce any other Method can attain to; thoſe Methods being always perplexed with various unknown Quantities, and the Solutions obtained by them terminating commonly in particular Caſes.

There are other Sorts of Series, *which tho' not properly infinite, yet are called Series, from the Regularity of the Terms whereof they are compoſed; thoſe Terms following one another with a certain uniformity, which is always to be defined. Of this nature is the Theorem given by Sir* Iſaac Newton, *in the fifth* Lemma *of the third Book of his* Principles, *for drawing a Curve through any given number of Points; of which the Demonſtration, as well as of other things belonging to the ſame Subject, may be deduced from the firſt Propoſition of his* Methodus Differentialis, *printed with ſome other of his Tracts, by the care of my Intimate Friend, and very ſkilful Mathematician, Mr.* W. Jones. *The abovementioned Theorem being very uſeful in ſumming up any number of Terms whoſe laſt Differences are equal, (ſuch as are the Numbers called Triangular, Pyramidal,* &c. *the Squares, the Cubes, or other Powers of Numbers in Arithmetic Progreſſion) I have ſhewn in many places of this Book how it might be applicable to theſe Caſes.*

After having dwelt ſome time upon various Queſtions depending on the general Principle of Combinations, as laid down in my Introduction, and upon ſome others depending on the Method of Infinite Series, I proceed to treat of the Method of Combinations properly ſo called, which I ſhew to be eaſily deducible from that more general Principle which had been before explained: Where it may be obſerved, that although the Caſes it is applied to are particular, yet the Way of Reaſoning, and the Conſequences derived from it, are general; that Method of

Arguing about generals by particular Examples, being in my opinion very convenient for eaſing the Reader's Imagination.

Having explained the common Rules of Combinations, and given a Theorem which may be of uſe for the Solution of ſome Problems relating to that Subject, I lay down a new Theorem, which is properly a contraction of the former, whereby ſeveral Queſtions of Chance are reſolved with wonderful eaſe, tho' the Solution might ſeem at firſt ſight to be of inſuperable difficulty.

It is by the Help of that Theorem ſo contracted, that I have been able to give a compleat Solution of the Problems of Pharaon *and* Baſſette, *which was never done before me: I own that ſome great Mathematicians had already taken the pains of calculating the advantage of the* Banker, *in any circumſtance either of Cards remaining in his Hands, or of any number of times that the Card of the* Ponte *is contained in the Stock: But ſtill the curioſity of the Inquiſitive remained unſatisfied; The Chief Queſtion, and by much the moſt difficult, concerning* Pharaon *or* Baſſette, *being, What it is that the Banker gets* per Cent. *of all the Money adventured at thoſe Games? which now I can certainly anſwer is very near Three* per Cent. *at* Pharaon, *and three fourths* per Cent. *at* Baſſette, *as may be ſeen in my 33d Problem, where the preciſe Advantage is calculated.*

In the 35th and 36th Problems, I explain a new ſort of Algebra, *whereby ſome Queſtions relating to Combinations are ſolved by ſo eaſy a Proceſs, that their Solution is made in ſome meaſure an immediate conſequence of the Method of Notation. I will not pretend to ſay that this new* Algebra *is abſolutely neceſſary to the Solving of thoſe Queſtions which I make to depend on it, ſince it appears that Mr.* Monmort, *Author of the* Analyſe des Jeux de Hazard, *and Mr.* Nicholas Bernoulli *have ſolved, by another Method, many of the caſes therein propoſed: But I hope I ſhall not be thought guilty of too much Confidence, if I aſſure the Reader, that the Method I have followed has a degree of Simplicity, not to ſay of Generality, which will hardly be attained by any other Steps than by thoſe I have taken.*

The 39th Problem, propoſed to me, amongſt ſome others, by the Honourable Mr. Francis Robartes, *I had ſolved in my tract* De menſura Sortis; *It relates, as well as the 35th and 36th, to the Method of Combinations, as is made to depend on the ſame Principle. When I began for the firſt time to attempt its Solution, I had nothing elſe to guide me but the common Rules of Combinations, ſuch as they had been delivered by Dr.* Wallis *and others; which when I endeavoured to apply, I was ſurprized to find that my Calculation ſwelled by degrees to an intolerable Bulk: For this reaſon I was forced to turn my Views another way, and to try* *whether the Solution I was ſeeking for might not be deduced from ſome eaſier conſiderations; whereupon I happily fell upon the Method I have been mentioning, which as it led me to a very great Simplicity in the Solution, ſo I look upon it to be an Improvement made to the Method of Combinations.*

The 40th Problem is the reverſe of the preceding; It contains a very remarkable Method of Solution, the Artifice of which conſiſts in changing an Arithmetic Progreſſion of Numbers into a Geometric one; this being always to be done when the Numbers are large, and their Intervals ſmall. I freely acknowledge that I have been indebted long ago for this uſeful Idea, to my much reſpected Friend, That Excellent Mathematician Dr. Halley, *Secretary to the* Royal Society, *whom I have ſeen practiſe the thing on another occaſion: For this and other Inſtructive Notions readily imparted to me, during an uninterrupted Friendſhip of five and Twenty years, I return him my very hearty Thanks.*

The 44th and 45th Problems, having in them a Mixture of the two Methods of Combinations and Infinite Series, may be propoſed for a pattern of Solution, in ſome of the moſt difficult caſes that may occur in the Subject of Chance, and on this occaſion I muſt do that Juſtice to Mr. Nicholas Bernoulli, *to own he had ſent me the Solution of thoſe Problems before mine was Publiſhed; which I had no ſooner received, but I communicated it to the* Royal Society, *and repreſented it as a Performance highly to be commended: Whereupon the Society order'd that his Solution ſhould be Printed; which was accordingly done ſome time after in the* Philoſophical Tranſactions, *Numb.* 341, *where mine was alſo inſerted.*

The Problems which follow relate chiefly to the Duration *of Play, or to the Method of determining what number of Games may probably be played out by two Adverſaries, before a certain number of Stakes agreed on between them be won or loſt on either ſide. This Subject affording a very great Variety of Curious Queſtions, of which every one has a degree of Difficulty peculiar to it ſelf, I thought it neceſſary to divide it into ſeveral diſtinct Problems, and to illuſtrate their Solution with proper Examples.*

Tho' theſe Queſtions may at firſt ſight ſeem to have a very great degree of difficulty, yet I have ſome reaſon to believe, that the Steps I have taken to come at their Solution, will eaſily be followed by thoſe who have a competent ſkill in Algebra, *and that the chief Method of proceeding therein will be underſtood by thoſe who are barely acquainted with the Elements of that Art.*

When I firſt began to attempt the general Solution of the Problem concerning the Duration of Play, there was nothing extant that could give me any light into that Subject; for altho' Mr. de Monmort, *in the firſt Edition of his Book, gives the Solution of this Problem, as limited to three Stakes to be won or loſt, and farther limited by the Suppoſition of an E-*

quality of Skill between the Adventurers; yet he having given no Demonſtration of his Solution, and the Demonſtration when diſcovered being of very little uſe towards obtaining the general Solution of the Problem, I was forced to try what my own Enquiry would lead me to which having been attended with Succeſs, the reſult of what I found was afterwards publiſhed in my Specimen *before mentioned.*

All the Problems which in my Specimen *related to the Duration of Play, have been kept entire in the following Treatiſe; but the Method of Solution has received ſome Improvements by the new Diſcoveries I have made concerning the Nature of thoſe Series which reſult from the Conſideration of the Subject; however, the Principles of that Method having been laid down in my* Specimen, *I had nothing now to do, but to draw the Conſequences that were naturally deducible from them.*

ADVERTISEMENT.

THE Author of this Work, by the failure of his Eye-ſight in extreme old age, was obliged to entruſt the Care of a new Edition of it to one of his Friends; to whom he gave a Copy of the former, with ſome marginal Corrections and Additions, in his own hand writing. To theſe the Editor has added a few more, where they were thought neceſſary: and has diſpoſed the whole in better Order; by reſtoring to their proper places ſome things that had been accidentally *miſplaced*, and by putting all the Problems concerning *Annuities* together; as they ſtand in the late *improved* Edition of the Treatiſe on that Subject. An *Appendix* of ſeveral uſeful Articles is likewiſe ſubjoined: the whole according to a Plan concerted with the Author, above a year before his death.

ERRATA.

Pag. 10 l. ult. for 445 read 455. p. 27 l. 4, for 10th read 12th Article. p. 29 l. 30, for xy read xz. p. 45 l. ult. for $\frac{1}{9}z^3$ read $\frac{1}{6}z^1$. p. 68. l. 6, for $d^5 + d^5$ read $d^4 + d^3$. p. 116 l. 26, for *Art.* 3d read *Art.* 4th. p. 179 l. 8 from the bottom, for XV. Prob. read *Corol.* to Prob. 19. p. 181 l. 1, for 18th read 19th. p. 187 l. 25, for 38 read 28. p. 192 l. 7. from bottom, for *aab* and *bba* read *aba* and *bab*. p. 205 l. 7, for $\frac{a^p}{\overline{a+b}|^p}$ read $\frac{b^p}{\overline{a+b}|^p}$. p. 238. l. 16, for *AFGz* read *AFz*.

THE

J. Mynde ſculp.

THE DOCTRINE OF CHANCES.

The INTRODUCTION.

1. THE Probability of an Event is greater or leſs, according to the number of Chances by which it may happen, compared with the whole number of Chances by which it may either happen or fail.

2. Wherefore, if we conſtitute a Fraction whereof the Numerator be the number of Chances whereby an Event may happen, and the Denominator the number of all the Chances whereby it may either happen or fail, that Fraction will be a proper deſignation of the Probability

bability of happening. Thus if an Event has 3 Chances to happen, and 2 to fail, the Fraction $\frac{3}{5}$ will fitly repreſent the Probability of its happening, and may be taken to be the meaſure of it.

The ſame thing may be ſaid of the Probability of failing, which will likewiſe be meaſured by a Fraction whoſe Numerator is the number of Chances whereby it may fail, and the Denominator the whole number of Chances, both for its happening and failing; thus the Probability of the failing of that Event which has 2 Chances to fail and 3 to happen will be meaſured by the Fraction $\frac{2}{5}$.

3. The Fractions which repreſent the Probabilities of happening and failing, being added together, their Sum will always be equal to Unity, ſince the Sum of their Numerators will be equal to their common Denominator: now it being a certainty that an Event will either happen or fail, it follows that Certainty, which may be conceived under the notion of an infinitely great degree of Probability, is fitly repreſented by Unity.

Theſe things will eaſily be apprehended, if it be conſidered, that the word Probability includes a double Idea; firſt, of the number of Chances whereby an Event may happen; ſecondly, of the number of Chances whereby it may either happen or fail.

If I ſay that I have three Chances to win any Sum of Money, it is impoſſible from that bare aſſertion to judge whether I am like to obtain it; but if I add that the number of Chances either to obtain it, or to miſs it, is five in all, from hence will enſue a compariſon between the Chances that favour me, and the whole number of Chances that are for or againſt me, whereby a true judgment will be formed of my Probability of ſucceſs: from whence it neceſſarily follows, that it is the comparative magnitude of the number of Chances to happen, in reſpect to the whole number of Chances either to happen or to fail, which is the true meaſure of Probability.

4. If upon the happening of an Event, I be intitled to a Sum of Money, my Expectation of obtaining that Sum has a determinate value before the happening of the Event.

Thus, if I am to have 10 *L.* in caſe of the happening of an Event which has an equal Probability of happening and failing, my Expectation before the happening of the Event is worth 5 *L.*: for I am preciſely in the ſame circumſtances as he who at an equal Play ventures 5 *L.* either to have 10, or to loſe his 5. Now he who ventures 5 *L.* at an equal Play, is poſſeſſor of 5 *L.* before the deciſion of the

Play; therefore my Expectation in the caſe above-mentioned muſt alſo be worth 5 *L.*

5. In all caſes, the Expectation of obtaining any Sum is eſtimated by multiplying the value of the Sum expected by the Fraction which repreſents the Probability of obtaining it.

Thus, if I have 3 Chances in 5 to obtain 100 *L.* I ſay that the preſent value of my Expectation is the product of 100 *L.* by the fraction $\frac{3}{5}$, and conſequently that my expectation is worth 60 *L.*

For ſuppoſing that an Event may equally happen to any one of 5 different Perſons, and that the Perſon to whom it happens ſhould in conſequence of it obtain the Sum of 100 *L.* it is plain that the right which each of them in particular has upon the Sum expected is $\frac{1}{5}$ of 100 *L.* which right is founded in this, that if the five Perſons concerned in the happening of the Event, ſhould agree not to ſtand the Chance of it, but to divide the Sum expected among themſelves, then each of them muſt have $\frac{1}{5}$ of 100 *L.* for his pretenſion. Now whether they agree to divide that ſum equally among themſelves, or rather chuſe to ſtand the Chance of the Event, no one has thereby any advantage or diſadvantage, ſince they are all upon an equal foot, and conſequently each Perſon's expectation is worth $\frac{1}{5}$ of 100 *L.* Let us ſuppoſe farther, that two of the five Perſons concerned in the happening of the Event, ſhould be willing to reſign their Chance to one of the other three; then the Perſon to whom thoſe two Chances are thus reſigned has now three Chances that favour him, and conſequently has now a right triple of that which he had before, and therefore his expectation is now worth $\frac{3}{5}$ of 100 *L.*

Now if we conſider that the fraction $\frac{3}{5}$ expreſſes the Probability of obtaining the Sum of 100 *L.*, and that $\frac{3}{5}$ of 100, is the ſame thing as $\frac{3}{5}$ multiplied by 100, we muſt naturally fall into this concluſion, which has been laid down as a principle, that the value of the Expectation of any Sum, is determined by multiplying the Sum expected by the Probability of obtaining it.

This manner of reaſoning, tho' deduced from a particular caſe, will eaſily be perceived to be general, and applicable to any other caſe.

COROLLARY.

From what precedes, it neceſſarily follows that if the Value of an Expectation be given, as alſo the Value of the thing expected, then dividing the firſt value by the ſecond, the quotient will expreſs the Probability of obtaining the Sum expected: thus if I have an Expectation worth 60 *l.* and that the Sum which I may obtain be worth 100 *l.* the Probability of obtaining it will be expreſt by the quotient of 60 divided by 100, that is by the fraction $\frac{60}{100}$ or $\frac{3}{5}$.

6. The Riſk of loſing any Sum is the reverſe of Expectation; and the true meaſure of it is, the product of the Sum adventured multiplied by the Probability of the Loſs.

7. Advantage or Diſadvantage in Play, reſults from the combination of the ſeveral Expectations of the Gameſters, and of their ſeveral Riſks.

Thus ſuppoſing that *A* and *B* play together, that *A* has depoſited 5 *L* and *B* 3 *l.* that the number of Chances which *A* has to win is 4, and the number of Chances which *B* has to win is 2, and that it were required in this circumſtance to determine the advantage or diſadvantage of the Adventurers, we may reaſon in this manner: Since the whole Sum depoſited is 8, and that the Probability which *A* has of getting it is $\frac{4}{6}$, it follows that the Expectation of *A* upon the whole Sum depoſited is $8 \times \frac{4}{6} = 5\frac{1}{3}$, and for the ſame reaſon the Expectation of *B* upon that whole Sum depoſited is $8 \times \frac{2}{6} = 2\frac{2}{3}$.

Now, if from the reſpective Expectations which the Adventurers have upon the whole ſum depoſited, be ſubtracted the particular Sums which they depoſit, that is their own Stakes, there will remain the Advantage or Diſadvantage of either, according as the difference is poſitive or negative.

And therefore if from $5\frac{1}{3}$, which is the Expectation of *A* upon the whole Sum depoſited, 5 which is his own Stake, be ſubtracted, there will remain $\frac{1}{3}$ for his advantage; likewiſe if from $2\frac{2}{3}$ which is the Expectation of *B*, 3 which is his own Stake be ſubtracted, there will remain $-\frac{1}{3}$, which being negative ſhews that his Diſadvantage is $\frac{1}{3}$.

Theſe concluſions may alſo be derived from another conſideration; for if from the Expectation which either Adventurer has upon the

Sum depoſited by his Adverſary, be ſubtracted the Riſk of what he himſelf depoſits, there will likewiſe remain his Advantage or Diſadvantage, according as the difference is poſitive or negative.

Thus in the preceding caſe, the Stake of B being 3, and the Probability which A has of winning it, being $\frac{4}{6}$, the Expectation of A upon that Stake is $3 \times \frac{4}{6} = 2$; moreover the Stake of A being 5, and the Probability of loſing it, being $\frac{2}{6}$, his Riſk ought to be eſtimated by $5 \times \frac{2}{6} = 1\frac{2}{3}$; wherefore, if from the Expectation 2, the Riſk $1\frac{2}{3}$ be ſubtracted, there will remain $\frac{1}{3}$ as before for the Advantage of A: and by the ſame way of proceeding, the Diſadvantage of B will be found to be $\frac{1}{3}$.

It is very carefully to be obſerved, that what is here called Advantage or Diſadvantage, and which may properly be called Gain or Loſs, is always eſtimated before the Event is come to paſs; and altho' it be not cuſtomary to call that Gain or Loſs which is to be derived from an Event not yet determined, nevertheleſs in the Doctrine of Chances, that appellation is equivalent to what in common diſcourſe is called Gain or Loſs.

For in the ſame manner as he who ventures a Guinea in an equal Game may, before the determination of the Play, be ſaid to be poſſeſſor of that Guinea, and may, in conſideration of that Sum, reſign his place to another; ſo he may be ſaid to be a Gainer or Loſer, who would get ſome Profit, or ſuffer ſome Loſs, if he would ſell his Expectation upon equitable terms, and ſecure his own Stake for a Sum equal to the Riſk of loſing it.

8. If the obtaining of any Sum requires the happening of ſeveral Events that are independent on each other, then the Value of the Expectation of that Sum is found by multiplying together the ſeveral Probabilities of happening, and again multiplying the product by the Value of the Sum expected.

Thus ſuppoſing that in order to obtain 90^{l} two Events muſt happen; the firſt whereof has 3 Chances to happen, and 2 to fail, the ſecond has 4 Chances to happen, and 5 to fail, and I would know the value of that Expectation; I ſay,

The Probability of the firſt's happening is $\frac{3}{5}$, the Probability of the ſecond's happening is $\frac{4}{9}$; now multiplying theſe two Probabilities together, the product will be $\frac{12}{45}$ or $\frac{4}{15}$; and this product being again multiplied by 90, the new product will be $\frac{360}{15}$ or 24, therefore that Expectation is worth 24 *L.*

The Demonſtration of this will be very eaſy, if it be conſider'd, that ſuppoſing the firſt Event had happened, then that Expectation depending now intirely upon the ſecond, would, before the determination of the ſecond, be found to be exactly worth $\frac{4}{9} \times 90$ *L.* or 40 *L.* (by Art. 5th) We may therefore look upon the happening of the firſt, as a condition of obtaining an Expectation worth 40 *L.* but the Probability of the firſt's happening has been ſuppoſed $\frac{3}{5}$, wherefore the Expectation ſought for is to be eſtimated by $\frac{3}{5} \times 40$, or by $\frac{3}{5} \times \frac{4}{9} \times 90$; that is, by the product of the two Probabilities of happening multiplied by the Sum expected.

And likewiſe, if an Expectation depends on the happening of one Event, and the failing of another, then its Value will be the product of the Probability of the firſt's happening by the Probability of the ſecond's failing, and of that again by the Value of the Sum expected.

And again, if an Expectation depends on the failing of two Events, the Rule will be the ſame; for that Expectation will be found by multiplying together the two Probabilities of failing, and multiplying that again by the Value of the Sum expected.

And the ſame Rule is applicable to the happening or failing of as many Events as may be aſſigned.

COROLLARY.

If we make abſtraction of the Value of the Sum to be obtained, the bare Probability of obtaining it, will be the product of the ſeveral Probabilities of happening, which evidently appears from this 8th Art. and from the Corollary to the 5th.

Hitherto, I have confined myſelf to the conſideration of Events independent; but for fear that, in what is to be ſaid afterwards, the terms independent or dependent might occaſion ſome obſcurity, it will be neceſſary, before I proceed any farther, to ſettle intirely the notion of thoſe terms.

Two Events are independent, when they have no connexion one with the other, and that the happening of one neither forwards nor obſtructs the happening of the other.

Two Events are dependent, when they are ſo connected together as that the Probability of either's happening is altered by the happening of the other.

In order to illuſtrate this, it will not be amiſs to propoſe the two following eaſy Problems.

1°. Suppoſe there is a heap of 13 Cards of one colour, and another heap of 13 Cards of another colour, what is the Probability that taking a Card at a venture out of each heap, I ſhall take the two Aces?

The Probability of taking the Ace out of the firſt heap is $\frac{1}{13}$: now it being very plain that the taking or not taking the Ace out of the firſt heap has no influence in the taking or not taking the Ace out of the ſecond; it follows, that ſuppoſing that Ace taken out, the Probability of taking the Ace out of the ſecond will alſo be $\frac{1}{13}$; and therefore, thoſe two Events being independent, the Probability of their both happening will be $\frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$.

2°. Suppoſe that out of one ſingle heap of 13 Cards of one colour, it ſhould be undertaken to take out the Ace in the firſt place, and then the Deux, and that it were required to aſſign the Probability of doing it; we are to conſider that altho' the Probability of the Ace's being in the firſt place be $\frac{1}{13}$, and that the Probability of the Deux's being in the ſecond place, would alſo be $\frac{1}{13}$, if that ſecond Event were conſidered in itſelf without any relation to the firſt; yet that the Ace being ſuppoſed as taken out at firſt, there will remain but 12 Cards in the heap, and therefore that upon the ſuppoſition of the Ace being taken out at firſt, the Probability of the Deux's being next taken will be alter'd, and become $\frac{1}{12}$; and therefore, we may conclude that thoſe two Events are dependent, and that the Probability of their both happening will be $\frac{1}{13} \times \frac{1}{12} = \frac{1}{156}$.

From whence it may be inferred, that the Probability of the happening of two Events dependent, is the product of the Probability of the happening of one of them, by the Probability which the other will have of happening, when the firſt is conſidered as having happened; and the ſame Rule will extend to the happening of as many Events as may be aſſigned.

9. But to determine, in the eaſieſt manner poſſible, the Probability of the happening of ſeveral Events dependent, it will be convenient to diſtinguiſh by thought the order of thoſe Events, and to ſuppoſe one of them to be the firſt, another to be the ſecond, and ſo on: which being done, the Probability of the happening of the firſt may

be looked upon as independent, the Probability of the happening of the ſecond, is to be determined from the ſuppoſition of the firſt's having happened, the Probability of the third's happening, is to be determined from the ſuppoſition of the firſt and ſecond having happened, and ſo on: then the Probability of the happening of them all will be the product of the Multiplication of the ſeveral Probabilities which have been determined in the manner preſcribed.

We had ſeen before how to determine the Probability of the happening or failing of as many Events independent as may be aſſigned; we have ſeen likewiſe in the preceding Article how to determine the Probability of the happening of as many Events dependent as may be aſſigned: but in the caſe of Events dependent, how to determine the Probability of the happening of ſome of them, and at the ſame time the Probability of the failing of ſome others, is a diſquiſition of a greater degree of difficulty; which for that reaſon will be more conveniently transferred to another place.

10. If I have ſeveral Expectations upon ſeveral Sums, it is very evident that my Expectation upon the whole is the Sum of the Expectations I have upon the particulars.

Thus ſuppoſe two Events ſuch, that the firſt may have 3 Chances to happen and 2 to fail, and the ſecond 4 Chances to happen and 5 to fail, and that I be intitled to $90^{L.}$ in caſe the firſt happens, and to another like Sum of 90^{L} in caſe the ſecond happens alſo, and that I would know the Value of my Expectation upon the whole: I ſay,

The Sum expected in the firſt caſe being $90^{L.}$ and the Probability of obtaining it being $\frac{3}{5}$, it follows that my Expectation on that account, is worth $90 \times \frac{3}{5} = 54$; and again the Sum expected in the ſecond caſe being 90, and the Probability of obtaining it being $\frac{4}{9}$, it follows that my Expectation of that ſecond Sum is worth $90 \times \frac{4}{9} = 40$; and therefore my Expectation upon the whole is worth $54^{L.} + 40^{L.} = 94^{L.}$

But if I am to have 90^{L} once for all for the happening of one or the other of the two afore-mentioned Events, the method of proceſs in determining the value of my Expectation will be ſomewhat altered: for altho' my Expectation of the firſt Event be worth $54^{L.}$ as it was in the preceding Example, yet I conſider that my Expectation of the ſecond will ceaſe upon the happening of the firſt, and that therefore this Expectation takes place only in caſe the firſt does happen to fail. Now the Probability of the firſt's failing is $\frac{2}{5}$; and ſuppoſing it has failed, then my Expectation will be 40; wherefore $\frac{2}{5}$ being the meaſure of the Probability of my obtaining an Expectation worth 40 *L.*, it follows that this Expectation (to eſtimate it before the time of the firſt's being determined) will be worth $40 \times \frac{2}{5} = 16$, and therefore my Expectation upon the whole is worth $54^{L.} + 16^{L.} = 70^{L.}$

If that which was called the ſecond Event be now conſidered as the firſt, and that which was called the firſt be now conſidered as the ſecond, the concluſion will be the ſame as before.

In order to make the preceding Rules familiar, it will be convenient to apply them to the Solution of ſome eaſy caſes, ſuch as are the following.

CASE I^{ſt.}

To find the Probability of throwing an Ace in two throws of one Die.

SOLUTION.

The Probability of throwing an Ace the firſt time is $\frac{1}{6}$; wherefore $\frac{1}{6}$ is the firſt part of the Probability required.

If the Ace be miſſed the firſt time, ſtill it may be thrown on the ſecond, but the Probability of miſſing it the firſt time is $\frac{5}{6}$, and the Probability of throwing it the ſecond time is $\frac{1}{6}$; wherefore the Probability of miſſing it the firſt time and throwing it the ſecond, is $\frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$: and this is the ſecond part of the Probability required, and therefore the Probability required is in all $\frac{1}{6} + \frac{5}{36} = \frac{11}{36}$.

To this caſe is analogous a queſtion commonly propoſed about throwing with two Dice either ſix or ſeven in two throws; which will be eaſily ſolved, provided it be known that Seven has 6 Chances to come up, and Six 5 Chances, and that the whole number of Chances in two Dice is 36: for the number of Chances for throwing ſix or ſeven being 11, it follows that the Probability of throwing either Chance the firſt time is $\frac{11}{36}$; but if both are miſſed the firſt time, ſtill either may be thrown the ſecond time; now the Probability of miſſing both the firſt time is $\frac{25}{36}$, and the Probability of throwing either of them the ſecond time is $\frac{11}{36}$; wherefore the Probability of miſſing both of them the firſt time, and throwing either of them the ſecond time, is $\frac{25}{36} \times \frac{11}{36} = \frac{275}{1296}$, and therefore the Probability required is $\frac{11}{36} + \frac{275}{1296} = \frac{671}{1296}$, and the Probability of the contrary is $\frac{625}{1296}$.

CASE II^d.

To find the Probability of throwing an Ace in three throws.

SOLUTION.

The Probability of throwing an Ace the firſt time is $\frac{1}{6}$, which is the firſt part of the Probability required.

If the Ace be miſſed the firſt time, ſtill it may be thrown in the two remaining throws; but the Probability of miſſing it the firſt time is $\frac{5}{6}$, and the Probability of throwing it in the two remaining times is (by Caſe 1^ſt) $= \frac{11}{36}$. And therefore the Probability of miſſing it the firſt time, and throwing it in the two remaining times is $\frac{5}{6} \times \frac{11}{36} = \frac{55}{216}$, which is the ſecond part of the Probability required; wherefore the Probability required will be $\frac{1}{6} + \frac{55}{216} = \frac{91}{216}$.

CASE III^d.

To find the Probability of throwing an Ace in four throws.

SOLUTION.

The Probability of throwing an Ace the firſt time is $\frac{1}{6}$, which is the firſt part of the Probability required.

If the Ace be miſſed the firſt time, of which the Probability is $\frac{5}{6}$, there remains the Probability of throwing it in three times, which (by Caſe 2^d) is $\frac{91}{216}$; wherefore the Probability of miſſing the Ace the firſt time, and throwing it in the three remaining times, is $= \frac{5}{6} \times \frac{91}{216} = \frac{445}{1296}$, which is the ſecond part of the Probability

bility required; and therefore the Probability required is, in the whole, $\frac{1}{6}+\frac{455}{1296}=\frac{671}{1296}$, and the Probability of the contrary $\frac{625}{1296}$.

It is remarkable, that he who undertakes to throw an Ace in four throws, has juſt the ſame Advantage of his adverſary, as he who undertakes with two Dice that ſix or ſeven ſhall come up in two throws, the odds in either caſe being 671 to 625: whereupon it will not be amiſs to ſhew how to determine eaſily the Gain of one Party from the Superiority of Chances he has over his adverſary, upon ſuppoſition that each ſtake is equal, and denominated by unity. For although this is a particular caſe of what has been explained in the 7th Article; yet as it is convenient to have the Rule ready at hand, and that it be eaſily remembered, I ſhall ſet it down here. Let therefore the odds be univerſally expreſſed by the ratio of a to b, then the reſpective Probabilities of winning being $\frac{a}{a+b}$, and $\frac{b}{a+b}$, the right of the firſt upon the Stake of the ſecond is $\frac{a}{a+b}\times 1$, and likewiſe the right of the ſecond upon the Stake of the firſt is $\frac{b}{a+b}\times 1$, and therefore the Gain of the firſt is $\frac{a-b}{a+b}\times 1$ or barely $\frac{a-b}{a+b}$: and conſequently the Gain of him who undertakes that ſix or ſeven ſhall come up in two throws, or who undertakes to fling an Ace in four throws, is $\frac{671-625}{671+625}=\frac{46}{1296}$, that is nearly $\frac{1}{28}$ part of his adverſary's Stake.

CASE IVth.

To find the Probability of throwing two Aces in two throws.

SOLUTION.

It is plain (by the 8th Art.) that the Probability required is $\frac{1}{6}\times\frac{1}{6}=\frac{1}{36}$.

CASE Vth.

To find the Probability of throwing two Aces in three throws.

SOLUTION.

If an Ace be thrown the firſt time, then it will only be required to throw it once in two throws; but the Probability of throwing it the firſt time is $\frac{1}{6}$, and the Probability of throwing it once in two throws (by the firſt caſe) is $\frac{11}{36}$: wherefore the Probability of throwing it the firſt time, and then throwing it once in the two remaining times is $\frac{1}{6} \times \frac{11}{36} = \frac{11}{216}$; and this is equal to the firſt part of the Probability required.

If the Ace be miſſed the firſt time, ſtill there remains the Probability of throwing it twice together, but the Probability of miſſing it the firſt time is $\frac{5}{6}$, and the Probability of throwing it twice together is (by the 4th Caſe) $= \frac{1}{36}$; therefore the Probability of both Events is $\frac{5}{6} \times \frac{1}{36} = \frac{5}{216}$; which is the ſecond part of the Probability required: therefore the whole Probability required is $\frac{11+5}{216} = \frac{16}{216}$.

CASE VIth.

To find the Probability of throwing two Aces in four throws.

SOLUTION.

If an Ace be thrown the firſt time, no more will be required than throwing it again in three throws; but the Probability of throwing an Ace the firſt time is $\frac{1}{6}$, and the Probability of throwing it in three times is $\frac{91}{216}$ (by the 2d Caſe;) wherefore the Probability of both happening is $\frac{1}{6} \times \frac{91}{216} = \frac{91}{1296} =$ 1ſt part of the Probability required.

If the Ace be miſſed the firſt time, ſtill there will remain the Probability of throwing two Aces in three throws; but the Probability of miſſing the Ace the firſt time is $\frac{5}{6}$, and the Probability of throwing it twice in three throws is $\frac{16}{216}$, (by the 5th Caſe;) wherefore the Probability of both together is $\frac{5}{6} \times \frac{16}{216} = \frac{80}{1296}$ $=$ 2d part of the Probability required: and therefore the Probability required $= \frac{91}{1296} + \frac{80}{1296} = \frac{171}{1296}$.

And, by the ſame way of reaſoning, we may gradually find the Probability of throwing an Ace as many times as ſhall be demanded, in any given number of throws.

If, inſtead of employing figures in the Solutions of the foregoing Caſes, we employ algebraic Characters, we ſhall readily perceive a moſt regular order in thoſe Solutions.

11. Let therefore a be the number of Chances for the happening of an Event, and b the number of Chances for its failing; then the Probability of its happening once in any number of Trials will be expreſſed by the Series $\frac{a}{a+b}+\frac{ab}{\overline{a+b}|^{2}}+\frac{abb}{\overline{a+b}|^{3}}+\frac{ab^{3}}{\overline{a+b}|^{4}}+\frac{ab^{4}}{\overline{a+b}|^{5}}+\frac{ab^{5}}{\overline{a+b}|^{6}}$, &c. which Series is to be continued to ſo many terms as are equal to the number of Trials given: thus if a be $=1$, $b=5$, and the number of Trials given $=4$, then the Probability required will be expreſſed by $\frac{1}{6}+\frac{5}{30}+\frac{25}{210}+\frac{125}{1296}=\frac{671}{1296}$.

The ſame things being ſuppoſed as before, the Probability of the Event's happening twice in any given number of Trials, will be expreſſed by the Series $\frac{aa}{\overline{a+b}|^{2}}+\frac{2aab}{\overline{a+b}|^{3}}+\frac{3aabb}{\overline{a+b}|^{4}}+\frac{4aab^{3}}{\overline{a+b}|^{5}}+\frac{5aab^{4}}{\overline{a+b}|^{6}}$, &c. which is to be continued to ſo many terms, wanting one, as is the number of Trials given; thus let us ſuppoſe $a=1$, $b=5$, and the number of Trials 8, then the Probability required will be expreſſed by $\frac{1}{30}+\frac{10}{210}+\frac{75}{1290}+\frac{500}{7770}+\frac{3125}{40050}+\frac{18750}{279936}+\frac{109375}{1679616}=\frac{663991}{1679616}$.

And again, the Probability of the Event's happening three times in any given number of Trials will be expreſſed by the Series $\frac{a^{3}}{\overline{a+b}|^{3}}+\frac{3a^{3}b}{\overline{a+b}|^{4}}+\frac{6a^{3}bb}{\overline{a+b}|^{5}}+\frac{10a^{3}b^{3}}{\overline{a+b}|^{6}}+\frac{15a^{3}b^{4}}{\overline{a+b}|^{7}}$, &c. which is to be continued to ſo many terms, wanting two, as is the number of terms given.

But all theſe particular Series may be comprehended under a general one, which is as follows.

Let a be the number of Chances, whereby an Event may happen, b the number of Chances whereby it may fail, l the number of times that the Event is required to be produced in any given number of Trials, and let n be the number of thoſe Trials; make $a+b=s$, then the Probability of the Event's happening l times in n Trials, will be expreſſed by the Series

$$\frac{a^{l}}{s^{l}}\times\overline{1+\frac{lb}{s}+\frac{l.l+1.bb}{1.2.ss}+\frac{l.l+1.l+2.b^{3}}{1.2.3.s^{3}}+\frac{l.l+1.l+2.l+3.b^{4}}{1.2.3.4.s^{4}}}$$

&c. which Series is to be continued to ſo many terms excluſive of the

It is to be noted here, and elſewhere, that the points here made uſe of, ſtand inſtead of the Mark of Multiplication ×.

the common multiplicator $\frac{a^l}{s^l}$ as are denoted by the number $n - l + 1$.

And for the ſame reaſon, the Probability of the contrary, that is of the Event's not happening ſo often as l times, making $n - l + 1 = p$, will be expreſſed by the Series $\frac{b^p}{s^p} \times \overline{1 + \frac{pa}{s} + \frac{p.\overline{p+1}.aa}{1.\ 2.\ ss} + \frac{p.\overline{p+1}.\overline{p+2}.a^3}{1.\ 2.\ 3.\ s^3} + \frac{p.\overline{p+1}.\overline{p+2}.\overline{p+3}.a^4}{1.\ 2.\ 3.\ 4.\ s^4}}$, which Series is to be continued to ſo many terms, excluſive of the common multiplicator, as are denoted by the number l.

Now the Probability of an Event's not happening being known, the Probability of its happening will likewiſe be known, ſince the Sum of thoſe two Probabilities is always equal to Unity; and therefore the ſecond Series, as well as the firſt, may be employed in determining the Probability of an Event's happening: but as the number of terms to be taken in the firſt is expreſſed by $n - l + 1$, and the number of terms to be taken in the ſecond is expreſſed by l, it will be convenient to uſe the firſt Series, if $n - l + 1$ be leſs than l, and to uſe the ſecond, if l be leſs than $n - l + 1$; or in other terms, to uſe the firſt or ſecond according as l is leſs or greater than $\frac{n+1}{2}$.

Thus, ſuppoſe an Event has 1 Chance to happen, and 35 to fail, and that it were required to aſſign the Probability of its happening in 24 Trials; then becauſe in this caſe $n = 24$ and $l = 1$, it is plain that 24 terms of the firſt Series would be requiſite to anſwer the Queſtion, and that one ſingle one of the ſecond will be ſufficient: and therefore, if in the ſecond Series we make $b = 35$, $a = 1$, and $l = 1$, the Probability of the Event's not happening once in 24 Trials, will be expreſſed by $\frac{35^{24}}{36^{24}} \times 1$, which by the help of Logarithms, we ſhall find nearly equivalent to the decimal fraction 0.50871; now this being ſubtracted from Unity, the remainder 0 49129 will expreſs the Probability required; and therefore the odds againſt the happening of the Event will be 50 to 49 nearly.

Again, ſuppoſe it be required to aſſign the Probability of the preceding Event's happening twice in 60 Trials; then becauſe $l = 2$, and $n = 60$, $n - l + 1$ will be $= 59$, which ſhews that 59 terms of the firſt Series would be required: but if we uſe the ſecond, then by reaſon of l being $= 2$, two of its terms will be ſufficient;

wherefore the two terms $\frac{35^{59}}{36^{59}} \times \overline{1 + \frac{59}{36}}$ will denote the Probability of the Event's not happening twice in 60 Trials. Now reducing this to a decimal fraction, it will be found equal to 0.5007, which being ſubtracted from Unity, the remainder 0.4993 will expreſs the Probability required; and therefore the odds againſt the Event's happening twice in 60 times will be very little more than 500 to 499.

It is to be obſerved of thoſe Series, that they are both derived from the ſame principle; for ſuppoſing two adverſaries A and B, contending about the happening of that Event which has every time a chances to happen, and b chances to fail, that the Chances a are favourable to A, and the Chances b to B, and that A ſhould lay a wager with B, that his Chances ſhall come up l times in n Trials: then by reaſon B lays a wager to the contrary, he himſelf undertakes that his own Chances ſhall, in the ſame number of Trials, come up $n - l + 1$ times; and therefore, if in the firſt Series, we change l into $n - l + 1$, and *vice verſâ*, and alſo write b for a, and a for b, the ſecond Series will be formed.

It will be eaſy to conceive how it comes to paſs, that if A undertakes to win l times in n Trials, his Adverſary B neceſſarily undertakes in the ſame number of Trials to win $n - l + 1$ times, if it be conſidered that A loſes his wager if he wins but $l - 1$ times; now if he wins but $l - 1$ times, then ſubtracting $l - 1$ from n, the remainder ſhews the number of times that B is to win, which therefore will be $n - l + 1$.

CASE VIIth.

To find the Probability of throwing one Ace, and no more, in four throws.

SOLUTION.

This Caſe ought carefully to be diſtinguiſhed from the fourth; for there it was barely demanded, without any manner of reſtriction, what the Probability was of throwing an Ace in 4 throws; now in this preſent caſe there is a reſtraint laid on that Event: for whereas in the former caſe, he who undertakes to throw an Ace deſiſts from throwing when once the Ace is come up; in this he obliges himſelf, after it is come up, to a farther Trial which is wholly againſt him, excepting the laſt throw of the four, after which there is no Trial; and therefore we ought from the unlimited Probability of the Ace's being thrown once in 4 throws, to ſubtract the Probability of its being thrown twice in that number of throws: now the firſt Probability is $\frac{671}{1296}$ (by the 3d Caſe, and the ſecond Probability is $\frac{171}{1296}$ (by the 6th Caſe,) from which it follows that the Probability required is $\frac{500}{1296}$, and the Probability of the contrary $\frac{796}{1296}$; and therefore the Odds againſt throwing one Ace and no more in 4 throws are 796 to 500, or 8 to 5 nearly: and the ſame method may be follow'd in higher caſes.

CASE VIIIth.

If A *and* B *play together, and that* A *wants but* 1 *Game of being up, and* B *wants* 2; *what are their reſpective Probabilities of winning the Set?*

SOLUTION.

It is to be conſidered that the Set will neceſſarily be ended in two Games at moſt, for if *A* wins the firſt Game, there is no need of any farther Trial; but if *B* wins it, then they will want each but 1 Game of being up, and therefore the Set will be determined by the ſecond Game: from whence it is plain that *A* wants only to win once in two Games, but that *B* wants to win twice together. Now ſuppoſing that *A* and *B* have an equal Chance to win a Game, then the Probability which *B* has of winning the firſt Game will be $\frac{1}{2}$, and conſequently the Probability of his winning twice together will be $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$; and therefore the Probability which *A* has of winning once in two Games will be $1 - \frac{1}{4} = \frac{3}{4}$, from whence it follows that the Odds of *A*ˢ winning are 3 to 1.

CASE IXth.

A *and* B *play together,* A *wants* 1 *Game of being up, and* B *wants* 2; *but the Chances whereby* B *may win a Game, are double to the number of Chances whereby* A *may win the ſame: 'tis required to aſſign the reſpective Probabilities of winning.*

SOLUTION.

It is plain that in this, as well as in the preceding caſe, *B* ought to win twice together; now ſince *B* has 2 Chances to win a Game, and *A* 1 Chance only for the ſame, the Probability which *B* has of winning a Game is $\frac{2}{3}$, and therefore the Probability of his winning twice together is $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$, and conſequently the Probability of *A*'s winning the Set is $1 - \frac{4}{9} = \frac{5}{9}$; from whence it follows that the Odds of *A*'s winning once, before *B* twice, are as 5 to 4.

REMARK.

Altho' the determining the preciſe Odds in queſtions of Chance requires calculation, yet ſometimes by a ſuperficial View of the queſtion, it may be poſſible to find that there will be an inequality in the Play. Thus in the preceding caſe wherein *B* has in every Game twice the number of Chances of *A*, if it be demanded whether *A* and *B* play upon the ſquare, it is natural to conſider that he who has a double number of Chances will at long run win twice as often as his Adverſary; but that the caſe is here otherwiſe, for *B* undertaking to win twice before *A* once, he thereby undertakes to win oftner than according to his proportion of Chances, ſince *A* has a right to expect to win once, and therefore it may be concluded that *B* has the diſadvantage: however, this way of arguing in general ought to be uſed with the utmoſt caution.

12. Whatever be the number of Games which *A* and *B* reſpectively want of being up, the Set will be concluded at the moſt in ſo many Games wanting one, as is the ſum of the Games wanted between them.

Thus ſuppoſe that *A* wants 3 Games of being up, and *B* 5; it is plain that the greateſt number of Games that *A* can win of *B* before the determination of the Play will be 2, and that the greateſt number which *B* can win of *A* before the determination of the Play will be 4; and therefore the greateſt number of Games that can be played between them before the determination of the Play will be 6: but ſuppoſing they have played ſix Games, the next Game will terminate the Play; and therefore the utmoſt number of Games that can be played between them will be 7, that is one Game leſs than the Sum of the Games wanted between them.

CASE Xth.

Suppoſing that A *wants* 3 *Games of being up, and* B *wants* 7; *but that the Chances which* A *and* B *reſpectively have for winning a Game are as* 3 *to* 5, *to find the reſpective Probabilities of winning the Set.*

SOLUTION.

By reaſon that the Sum of the Games wanted between *A* and *B* is 10, it is plain by the preceding Paragraph that the Set will be concluded in 9 Games at moſt, and that therefore *A* undertakes out of 9 Games to win 3, and *B*, out of the ſame number, to win 7; now ſuppoſing that the firſt general Theorem laid down in the 11th Art. is particularly adapted to repreſent the Probability of *A*^{s} winning, then $l = 3$; and becauſe n repreſents the number of Games in which the Set will be concluded, $n = 9$; but the number of terms to be uſed in the firſt Theorem being $= n - l + 1 = 7$, and the number of terms to be uſed in the ſecond Theorem being $= l = 3$, it will be more convenient to uſe the ſecond, which will repreſent the Probability of *B*^{s} winning. Now that ſecond Theorem being applied to the caſe of n being $= 9$, $l = 3$, $a = 3$, $b = 5$, the Probability which *B* has of winning the Set will be expreſſed by $\frac{5^7}{8^7} \times \overline{1 + \frac{21}{8} + \frac{252}{64}} = \frac{5^7}{8^9} \times 484 = 0.28172$ nearly; and therefore ſubtracting this from Unity, there will remain the Probability which *A* has of winning the ſame, which will be $= 0.71828$: and conſequently the Odds of *A*^{s} winning the Set will be 71828 to 28172, or very near as 23 to 9.

The ſame Principles explained in a different and more general way.

Altho' the principles hitherto explained are a ſufficient introduction to what is to be ſaid afterwards; yet it will not be improper to reſume ſome of the preceding Articles, and to ſet them in a new light: it frequently happening that ſome truths, when repreſented to the mind under a particular Idea, may be more eaſily apprehended than when repreſented under another.

13. Let us therefore imagine a Die of a given number of equal faces, let us likewiſe imagine another Die of the ſame or any other number of equal faces; this being ſuppoſed, I ſay that the number of all the variations which the two Dice can undergo will be obtained by multiplying the number of faces of the one, by the number of faces of the other.

In order to prove this, and the better to fix the imagination, let us take a particular caſe: Suppoſe therefore that the firſt Die contains 8 faces, and the ſecond 12; then ſuppoſing the firſt Die to ſtand ſtill upon one of its faces, it is plain that in the mean time the ſecond Die may revolve upon its 12 faces; for which reaſon, there will be upon that ſingle ſcore 12 variations: let us now ſuppoſe that the firſt Die ſtands upon another of its faces, then whilſt that Die ſtands ſtill, the ſecond Die may revolve again upon its 12 faces; and ſo on, till the faces of the firſt Die have undergone all their changes: from whence it follows, that in the two Dice, there will be as many times 12 Chances as there are faces in the firſt Die; but the number of faces in the firſt Die has been ſuppoſed 8, wherefore the number of Variations or Chances of the two Dice will be 8 times 12, that is 96: and therefore it may be univerſally concluded, that the number of all the variations of two Dice will be the product of the multiplication of the number of faces of one Die, by the number of faces of the other.

14. Let us now imagine that the faces of each Die are diſtinguiſhed into white and black, that the number of white faces upon the firſt is A, and the number of black faces B, and alſo that the number of white faces upon the ſecond is a, and the number of black faces b; hence it will follow by the preceding Article, that multiplying $A + B$ by $a + b$, the product $Aa + Ab + Ba + Bb$, will exhibit all the Variations of the two Dice: Now let us ſee what each of theſe four parts ſeparately taken will repreſent.

1°. It is plain, that in the ſame manner as the product of the multiplication of the whole number of faces of the firſt Die, by the whole number of faces of the ſecond, expreſſes all the variations of the two Dice; ſo likewiſe the multiplication of the number of the white faces of the firſt Die, by the number of the white faces of the ſecond, will expreſs the number of variations whereby the two Dice may exhibit two white faces: and therefore, that number of Chances will be repreſented by Aa.

2°. For the ſame reaſon, the multiplication of the number of white faces upon the firſt Die, by the number of black faces upon the ſecond, will repreſent the number of all the Chances whereby a white face of the firſt may be joined with a black face of the ſecond; which number of Chances will therefore be repreſented by Ab.

3°. The multiplication of the number of white faces upon the ſecond, by the number of black faces upon the firſt, will expreſs the number of all the Chances whereby a white face of the ſecond may be joined with a black face of the firſt; which number of Chances will therefore be repreſented by aB.

4°. The multiplication of the number of black faces upon the firſt, by the number of black faces upon the ſecond, will expreſs the number of all the Chances whereby a black face of the firſt may be joined with a black face of the ſecond; which number of Chances will therefore be repreſented by Bb.

And therefore we have explained the proper ſignification and uſe of the ſeveral parts Aa, Ab, Ba, Bb ſingly taken.

But as theſe parts may be connected together ſeveral ways, ſo the Sum of two or more of any of them will anſwer ſome queſtion of Chance: for inſtance, ſuppoſe it be demanded, what is the number of Chances, with the two Dice above-mentioned, for throwing a white face? it is plain that the three parts $Aa + Ab + Ba$ will anſwer the queſtion; ſince every one of thoſe parts comprehends a caſe wherein a white face is concerned.

It may perhaps be thought that the firſt term Aa is ſuperfluous, it denoting the number of Variations whereby two white faces can be thrown; but it will be eaſy to be ſatisfied of the neceſſity of taking it in: for ſuppoſing a wager depending on the throwing of a white face, he who throws for it, is reputed a winner, whenever a white face appears, whether one alone, or two together, unleſs it be expreſly ſtipulated that in caſe he throws two, he is to loſe his wager; in which latter caſe the two terms $Ab + Ba$ would repreſent all his Chances.

If now we imagine a third Die having upon it a certain number of white faces repreſented by α, and likewiſe a certain number of black faces repreſented by β, then multiplying the whole variation of Chances of the two preceding Dice *viz.* $Aa + Ab + Ba + Bb$ by the whole number of faces $\alpha + \beta$ of the third Die, the product $Aa\alpha + Ab\alpha + Ba\alpha + Bb\alpha + Aa\beta + Ab\beta + Ba\beta + Bb\beta$ will exhibit the number of all the Variations which the three Dice can undergo.

Examining the ſeveral parts of this new product, we may eaſily perceive that the firſt term $Aa\alpha$ repreſents the number of Chances for throwing three white faces, that the ſecond term $Ab\alpha$ repreſents the number of Chances whereby both the firſt and third Die may exhibit a white face, and the ſecond Die a black one; and that the reſt of the terms have each their particular properties, which are diſcovered by bare inſpection.

It may alſo be perceived, that by joining together two or more of thoſe terms, ſome queſtion of Chance will thereby be anſwered: for inſtance, if it be demanded what is the number of Chances for throwing two white faces and a black one? it is plain that the three terms $Ab\alpha$, $Ba\alpha$, $Aa\beta$ taken together will exhibit the number of Chances required, ſince in every one of them there is the expreſſion of two white faces and a black one; and therefore if there be a wager depending on the throwing two white faces and a black one, he who undertakes that two white faces and a black one ſhall come up, has for him the Odds of $Ab\alpha + Ba\alpha + Aa\beta$ to $Aa\alpha + Bb\alpha + Ab\beta + Ba\beta + Bb\beta$; that is, of the three terms that include the condition of the wager, to the five terms that include it not.

When the number of Chances that was required has been found, then making that number the Numerator of a fraction, whereof the Denominator is the whole number of variations which all the Dice can undergo, that fraction will expreſs the Probability of the Event; as has been ſhewn in the firſt Article.

Thus if it was demanded what the Probability is, of throwing three white faces with the three Dice above-mentioned, that Probability will be expreſſed by the fraction $\frac{Aa\alpha}{Aa\alpha + Ab\alpha + Ba\alpha + Aa\beta + Bb\alpha + Ab\beta + Ba\beta + Bb\beta}$.

But it is to be obſerved, that in writing the Denominator, it will be convenient to expreſs it by way of product, diſtinguiſhing the ſeveral multiplicators whereof it is compounded; thus in the preceding caſe the Probability required will be beſt expreſſed as follows, $\frac{Aa\alpha}{\overline{A+B} \times \overline{a+b} \times \overline{\alpha+\beta}}$.

If the preceding fraction be conceived as the product of the three fractions $\frac{A}{A+B} \times \frac{a}{a+b} \times \frac{\alpha}{\alpha+\beta}$, whereof the firſt expreſſes the Probability of throwing a white face with the firſt Die; the ſecond the Probability of throwing a white face with the ſecond Die, and the third the Probability of throwing a white face with the third Die; then will again appear the truth of what has been demonſtrated in the 8th *Art.* and its *Corollary*, *viz.* that the Probability of the happening of ſeveral Events independent, is the product of all the particular Probabilities whereby each particular Event may be produced; for altho' the caſe here deſcribed be confined to three Events, it is plain that the Rule extends itſelf to any number of them.

Let us reſume the caſe of two Dice, wherein we did ſuppoſe that the number of white faces upon one Die was expreſſed by A, and the number of black faces by B, and alſo that the number of white faces upon the other was expreſſed by a, and the number of black faces by b, which gave us all the variations $Aa + Ab + aB + Bb$;

and let us imagine that the number of the white and black faces is reſpectively the ſame upon both Dice: wherefore $A = a$, and $B = b$, and conſequently inſtead of $Aa + Ab + aB + Bb$, we ſhall have $aa + ab + ab + bb$, or $aa + 2ab + bb$; but in the firſt caſe $Ab + aB$ did expreſs the number of variations whereby a white face and a black one might be thrown, and therefore $2ab$ which is now ſubſtituted in the room of $Ab + aB$ does expreſs the number of variations, whereby with two Dice of the ſame reſpective number of white and black faces, a white face and a black one may be thrown.

In the ſame manner, if we reſume the general caſe of three Dice, and examine the number of variations whereby two white faces and a black one may be thrown, it will eaſily be perceived that if the number of white and black faces upon each Die are reſpectively the ſame, then the three parts $Ab\alpha + Ba\alpha + Aa\beta$ will be changed into $aba + baa + aab$, or $3aab$, and that therefore $3aab$, which is one term of the Binomial $a + b$ raiſed to its Cube, will expreſs the number of variations whereby three Dice of the ſame kind would exhibit two white faces and a black one.

15. From the preceding conſiderations, this general Rule may be laid down, *viz.* that if there be any number of Dice of the ſame kind, all diſtinguiſhed into white and black faces, that n be the number of thoſe Dice, a and b the reſpective numbers of white and black faces upon each Die, and that the Binomial $a + b$ be raiſed to the power n; then 1°, the firſt term of that power will expreſs the number of Chances whereby n white faces may be thrown; 2°, that the ſecond term will expreſs the number of Chances whereby $n - 1$ white faces and 1 black face may be thrown; 3°, that the third term will expreſs the number of Chances whereby $n - 2$ white faces and 2 black ones may be thrown; and ſo on for the reſt of the terms.

Thus, for inſtance, if the Binomial $a + b$ be raiſed to its 6th power, which is $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$; the firſt term a^6 will expreſs the number of Chances for throwing 6 white faces; the ſecond term $6a^5b$ will expreſs the number of Chances for throwing 5 white and 1 black; the third term $15a^4b^2$ will expreſs the number of Chances for throwing 4 white and 2 black; the fourth $20a^3b^3$ will expreſs the number of Chances for throwing 3 white and 3 black; the fifth $15a^2b^4$ will expreſs the number of Chances for throwing two white and 4 black; the ſixth $6ab^3$ will expreſs the number of Chances for 2 white and 4 black; laſtly, the ſeventh b^6 will expreſs the number of Chances for 6 black.

And therefore having raiſed the Binomial $a+b$ to any given power, we may by bare inſpection determine the property of any one term belonging to that power, by only obſerving the Indices wherewith the quantities a and b are affected in that term, ſince the reſpective numbers of white and black faces are repreſented by thoſe Indices.

The better to compare the conſequences that may be derived from the conſideration of the Binomial $a+b$ raiſed to a power given, with the method of Solution that hath been explained before; let us reſume ſome of the preceding queſtions, and ſee how the Binomial can be applied to them.

Suppoſe therefore that the Probability of throwing an Ace in four throws with a common Die of ſix faces be demanded.

In order to anſwer this, it muſt be conſidered that the throwing of one Die four times ſucceſſively, is the ſame thing as throwing four Dice at once; for whether the ſame Die is uſed four times ſucceſſively, or whether a different Die is uſed in each throw, the Chance remains the ſame; and whether there is a long or a ſhort interval between the throwing of each of theſe four different Dice, the Chance remains ſtill the ſame; and therefore if four Dice are thrown at once, the Chance of throwing an Ace will be the ſame as that of throwing it with one and the ſame Die in four ſucceſſive throws.

This being premiſed, we may transfer the notion that was introduced concerning white and black faces, in the Dice, to the throwing or miſſing of any point or points upon thoſe Dice; and therefore in the preſent caſe of throwing an Ace with four Dice, we may ſuppoſe that the Ace in each Die anſwer to one white face, and the reſt of the points to five black faces, that is, we may ſuppoſe that $a=1$, and $b=5$; and therefore, having raiſed $a+b$ to its fourth power, which is $a^4+4a^3b+6a^2b^2+4ab^3+b^4$, every one of the terms wherein a is perceived will be a part of the number of Chances whereby an Ace may be thrown. Now there are four of thoſe parts into which a enters, *viz.* $a^4+4a^3b+6a^2b^2+4ab^3$, and therefore having made $a=1$, and $b=5$, we ſhall have $1+20+150+500=671$ to expreſs the number of Chances whereby an Ace may be thrown with four Dice, or an Ace thrown in four ſucceſſive throws of one ſingle Die: but the number of all the Chances is the fourth power of $a+b$, that is the fourth power of 6, which is 1296; and therefore the Probability required is meaſured by the fraction $\frac{671}{1296}$, which is conformable to the reſolution given in the 3d caſe of the queſtions belonging to the 10th Art.

It is to be obſerved, that the Solution would have been ſhorter, if inſtead of inquiring at firſt into the Probability of throwing an Ace in four throws, the Probability of its contrary, that is the Probability of miſſing the Ace four times ſucceſſively, had been inquired into; for ſince this caſe is exactly the ſame as that of miſſing all the Aces with four Dice, and that the laſt term b^4 of the Binomial $a+b$ raiſed to its fourth power expreſſes the number of Chances whereby the Ace may fail in every one of the Dice; it follows, that the Probability of that failing is $\frac{b^4}{\overline{a+b}|^4} = \frac{625}{1296}$, and therefore the Probability of not failing, that is of throwing an Ace in four throws, is $1 - \frac{625}{1296} = \frac{1296-625}{1296} = \frac{671}{1296}$.

From hence it follows, that let the number of Dice be what it will, ſuppoſe n, then the laſt term of the power $\overline{a+b}|^n$, that is b^n, will always repreſent the number of Chances whereby the Ace may fail n times, whether the throws be conſidered as ſucceſſive or cotemporary: Wherefore $\frac{b^n}{\overline{a+b}|^n}$ is the Probability of that failing; and conſequently the Probability of throwing an Ace in a number of throws expreſſed by n, will be $1 - \frac{b^n}{\overline{a+b}|^n} = \frac{\overline{a+b}|^n - b^n}{\overline{a+b}|^n}$.

Again, ſuppoſe it be required to aſſign the Probability of throwing with one ſingle Die two Aces in four throws, or of throwing at once two Aces with four Dice: the queſtion will be anſwered by help of the Binomial $a+b$ raiſed to its fourth power, which being $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, the three terms $a^4 + 4a^3b + 6a^2b^2$ wherein the Indices of a equal or exceed the number of times that the Ace is to be thrown, will denote the number of Chances whereby two Aces may be thrown; wherefore having interpreted a by 1, and b by 5, the three terms above-written will become $1 + 20 + 150 = 171$, but the whole number of Chances, *viz.* $\overline{a+b}|^4$ is in this caſe $= 1296$, and therefore the Probability of throwing two Aces in four throws will be meaſured by the fraction $\frac{171}{1296}$.

But if we chuſe to take at firſt the Probability of the contrary, it is plain that out of the five terms that the fourth power of $a+b$ conſiſts of, the two terms $4ab^3 + b^4$; in the firſt of which a enters but once, and in the ſecond of which it enters not, will expreſs the number of Chances that are contrary to the throwing of two Aces; which number of Chances will be found equal to $500 + 625 = 1125$. And therefore the Probability of not throwing two Aces in four throws will be $\frac{1125}{1296}$: from whence may be deduced the Probability of doing it, which therefore will be $1 - \frac{1125}{1296} = \frac{1296-1125}{1296} = \frac{171}{1296}$ as it was found in the preceding paragraph; and this agrees with the Solution of the ſixth Caſe to be ſeen in the 10th Article.

Univerſally, the laſt term of any power $\overline{a+b}|^n$ being b^n, and the laſt but one being nab^{n-1}, in neither of which a^2 enters, it follows that the two laſt terms of that power expreſs the number of Chances that are contrary to the throwing of two Aces, in any number of throws denominated by n; and that the Probability of throwing two Aces will be $1 - \frac{nab^{n-1}+b^n}{\overline{a+b}|^n} = \frac{\overline{a+b}|^n - nab^{n-1} - b^n}{\overline{a+b}|^n}$.

And likewiſe, in the three laſt terms of the power $\overline{a+b}|^n$, every one of the Indices of a will be leſs than 3, and conſequently thoſe three laſt terms will ſhew the number of Chances that are contrary to the throwing of an Ace three times in any number of Trials denominated by n: and the ſame Rule will hold perpetually.

And theſe concluſions are in the ſame manner applicable to the happening or failing of any other ſort of Event in any number of times, the Chances for happening and failing in any particular Trial being reſpectively repreſented by a and b.

16. Wherefore we may lay down this general Maxim; that ſuppoſing two Adverſaries A and B contending about the happening of an Event, whereof A lays a wager that the Event will happen l times in n Trials, and B lays to the contrary, and that the number of Chances whereby the Event may happen in any one Trial are a, and the number of Chances whereby it may fail are b, then ſo many of the laſt terms of the power $\overline{a+b}|^n$ expanded, as are repreſented by l, will ſhew the number of Chances whereby B may win his wager.

Again, B laying a wager that A will not win l times, does the ſame thing in effect as if he laid that A will not win above $l-1$ times; but the number of winnings and loſings between A and B is n by hypotheſis, they having been ſuppoſed to play n times, and therefore ſubtracting $l-1$ from n, the remainder $n-l+1$ will ſhew that B himſelf undertakes to win $n-l+1$ times; let this remainder be called p, then it will be evident that in the ſame manner as the laſt terms of the power $\overline{a+b}|^n$ expanded, *viz.* $b^n +$

$nab^{n-1} + \frac{n}{1} \times \frac{n-1}{2} a^2b^{n-2}$, &c. the number whereof is l, do expreſs the number of Chances whereby B may be a winner, ſo the firſt terms $a^n + na^{n-1}b + \frac{n}{1} \times \frac{n-1}{2} a^{-2}b^2$, &c. the number whereof is p, do expreſs the number of Chances whereby A may be a winner.

17. If A and B being at play, reſpectively want a certain number of Games l and p of being up, and that the reſpective Chances they have for winning any one particular Game be in the proportion of a to b; then raiſing the Binomial $a + b$ to a power whoſe Index ſhall be $l + p - 1$, the number of Chances whereby they may reſpectively win the Set, will be in the ſame proportion as the Sum of ſo many of the firſt terms as are expreſſed by p, to the Sum of ſo many of the laſt terms as are expreſſed by l.

This will eaſily be perceived to follow from what was ſaid in the preceding Article: for when A and B reſpectively undertook to win l Games and p Games, we have proved that if n was the number of Games to be played between them, then p was neceſſarily equal to $n - l + 1$, and therefore $l + p = n + 1$, and $n = l + p - 1$; and conſequently the power to which $a + b$ is to be raiſed will be $l + p - 1$.

Thus ſuppoſing that A wants 3 Games of being up, and B 7, that their proportion of Chances for winning any one Game are reſpectively as 3 to 5, and that it were required to aſſign the proportion of Chances whereby they may win the Set; then making $l = 3$, $p = 7$, $a = 3$, $b = 5$, and raiſing $a + b$ to the power denoted by $l + p - 1$, that is in this caſe to the 9th power, the Sum of the firſt ſeven terms will be to the Sum of the three laſt, in the proportion of the reſpective Chances whereby they may win the Set.

Now it will be ſufficient in this caſe to take the Sum of the three laſt terms; for ſince that Sum expreſſes the number of Chances whereby B may win the Set, then it being divided by the 9th power of $a + b$, the quotient will exhibit the Probability of his winning; and this Probability being ſubtracted from Unity, the remainder will expreſs the Probability of A's winning: but the three laſt terms of the Binomial $a + b$ raiſed to its 9th power are $b^9 + 9ab^8 + 36aab^7$, which being converted into numbers make the Sum 37812500, and the 9th power of $a + b$ is 134217728, and therefore the Probability of B's winning will be expreſſed by the fraction $\frac{37812500}{134217728} = \frac{9453125}{33554432}$; let this be ſubtracted from Unity, then the remainder

$\frac{24101307}{33554432}$ will expreſs the Probability of *A*'s winning; and therefore the Odds of *A*'s being up before *B*, are in the proportion of 24101307 to 9453125, or very near as 23 to 9: which agrees with the Solution of the 10th Caſe included in the 10th Article.

In order to compleat the compariſon between the two Methods of Solution which have been hitherto explained, it will not be improper to propoſe one caſe more.

Suppoſe therefore it be required to aſſign the Probability of throwing one Ace and no more, with four Dice thrown at once.

It is viſible that if from the number of Chances whereby one Ace or more may be thrown, be ſubtracted the number of Chances whereby two Aces or more may be thrown, there will remain the number of Chances for throwing one Ace and no more; and therefore having raiſed the Binomial $a+b$ to its fourth power, which is $a^4+4a^3b+6a^2b^2+4ab^3+b^4$, it will plainly be ſeen that the four firſt terms expreſs the number of Chances for throwing one Ace or more, and that the three firſt terms expreſs the number of Chances for throwing two Aces or more; from whence it follows that the ſingle term $4ab^3$ does alone expreſs the number of Chances for throwing one Ace and no more, and therefore the Probability required will be $\frac{4ab^3}{\overline{a+b}|^4}=\frac{500}{1296}=\frac{125}{324}$: which agrees with the Solution of the 7th Caſe given in the 10th Article.

This Concluſion might alſo have been obtained another way: for applying what has been ſaid in general concerning the property of any one term of the Binomial $a+b$ raiſed to a power given, it will thereby appear that the term $4ab^3$ wherein the indices of a and b are reſpectively 1 and 3, will denote the number of Chances whereby of two contending parties *A* and *B*, the firſt may win once, and the other three times. Now *A* who undertakes that he ſhall win once and no more, does properly undertake that his own Chance ſhall come up once, and his adverſary's three times; and therefore the term $4ab^3$ expreſſes the number of Chances for throwing one Ace and no more.

In the like manner, if it be required to aſſign the Chances for throwing a certain number of Aces, and it be farther required that there ſhall not be above that number, then one ſingle term of the power $\overline{a+b}|^n$ will always anſwer the queſtion.

But to find that term as expeditiouſly as poſſible, ſuppoſe n to be the number of Dice, and l the preciſe number of Aces to be thrown; then if l be leſs than $\frac{1}{2}n$, write as many terms of the

Series $\frac{n}{1}$, $\frac{n-1}{2}$, $\frac{n-2}{3}$, $\frac{n-3}{4}$, $\frac{n-4}{5}$, &c. as there are Units in l; or if l be greater than $\frac{1}{2}n$, write as many of them as there are Units in $\frac{1}{2}n - l$; then let all thoſe terms be multiplied together, and the product be again multiplied by $a^l b^{n-l}$; and this laſt product will exhibit the term expreſſing the number of Chances required.

Thus if it be required to aſſign the number of Chances for throwing preciſely three Aces, with ten Dice; here l will be $= 3$, and $n = 10$. Now becauſe l is leſs than $\frac{1}{2}n$, let ſo many terms be taken of the Series $\frac{n}{1}$, $\frac{n-1}{2}$, $\frac{n-2}{3}$, $\frac{n-3}{4}$, &c. as there are Units in 3, which terms in this particular caſe will be $\frac{10}{1}$, $\frac{9}{2}$, $\frac{8}{3}$; let thoſe terms be multiplied together, the product will be 120; let this product be again multiplied by $a^l b^{n-l}$, that is (a being $= 1$, $b = 9$, $l = 3$, $n = 10$) by 6042969, and the new product will be 725156280, which conſequently exhibits the number of Chances required. Now this being divided by the 10th power of $a + b$, that is, in this caſe, by 10000000000, the quotient 0.0725156280 will expreſs the Probability of throwing preciſely three Aces with ten Dice; and this being ſubtracted from Unity, the remainder 0.9274843720 will expreſs the Probability of the contrary; and therefore the Odds againſt throwing three Aces preciſely with ten Dice are 9274843720, to 725156280, or nearly as 64 to 5.

Although we have ſhewn above how to determine univerſally the Odds of winning, when two Adverſaries being at play, reſpectively want certain number of Games of being up, and that they have any given proportion of Chances for winning any ſingle Game; yet I have thought it not improper here to annex a ſmall Table, ſhewing thoſe Odds, when the number of Games wanting, does not exceed ſix, and that the Skill of the Contenders is equal.

Games wanting.	Odds of winning.	Games wanting.	Odds of winning.	Games wanting.	Odds of winning.
1, 2	3, 1	2, 3	11, 5	3, 5	99, 29
1, 3	7, 1	2, 4	26, 6	3, 6	219, 37
1, 4	15, 1	2, 5	57, 7	4, 5	163, 93
1, 5	31, 1	2, 6	120, 8	4, 6	382, 130
1, 6	63, 1	3, 4	42, 22	5, 6	638, 386

Before I put an end to this Introduction, it will not be improper to ſhew how ſome operations may often be contracted by barely introducing one ſingle Letter, inſtead of two or three, to denote the Probability of the happening of one Event.

18. Let therefore x denote the Probability of one Event; y, the Probability of a ſecond Event; z, the Probability of the happening of a third Event: then it will follow, from what has been ſaid in the beginning of this Introduction, that $1-x$, $1-y$, $1-z$ will repreſent the reſpective Probabilities of their failing.

This being laid down, it will be eaſy to anſwer the Queſtions of Chance that may ariſe concerning thoſe Events.

1°. Let it be demanded, what is the Probability of the happening of them all; it is plain by what has been demonſtrated before, that the anſwer will be denoted by xyz.

2°. If it is inquired, what will be the Probability of their all failing; the anſwer will be $\overline{1-x} \times \overline{1-y} \times \overline{1-z}$, which being expanded by the Rules of Multiplication would be $1-x-y-z+xy+xz+yz-xyz$; but the firſt expreſſion is more eaſily adapted to Numbers.

3°. Let it be required to aſſign the Probability of ſome one of them or more happening; as this queſtion is exactly equivalent to this other, what is the Probability of their not all failing? the anſwer will be $1-\overline{1-x} \times \overline{1-y} \times \overline{1-z}$, which being expanded will become $x+y+z-xy-xz-yz+xyz$.

4°. Let it be demanded what is the Probability of the happening of the firſt and ſecond, and at the ſame time of the failing of the third, the Queſtion is anſwered by barely writing it down algebraically; thus, $xy \times \overline{1-z}$, or $xy-xyz$: and for the ſame reaſon the Probability of the happening of the firſt and third, and the failing of the ſecond, will be $xz \times \overline{1-y}$ or $xy-xyz$: and for the ſame reaſon again, the Probability of the happening of the ſecond and third, and the failing of the firſt, will be $yz \times \overline{1-x}$, or $yz-xyz$. And the Sum of thoſe three Probabilities, *viz.* $xy+xz+yz-3xyz$. will expreſs the Probability of the happening of any two of them, but of no more than two.

5°. If it be demanded what is the Probability of the happening of the firſt, to the excluſion of the other two, the anſwer will be $x \times \overline{1-y} \times \overline{1-z}$, or $x-xy-xz+xyz$; and in the ſame manner, the Probability of the happening of the ſecond to the excluſion of the other two, will be $y-xy-yz+xyz$; and again, the Probability of the happening of the third, to the excluſion of the other two, will be $z - xz - yz + xyz$, and the Sum of all theſe Probabilities together, *viz.* $x + y + z - 2xy - 2xz - 2yz + 3xyz$ will expreſs the Probability of the happening of any one of them, and of the failing of the other two: and innumerable caſes of the ſame nature, belonging to any number of Events, may be ſolved without any manner of trouble to the imagination, by the mere force of a proper Notation.

REMARK.

I. When it is required to ſum up ſeveral Terms of a high Power of the Binomial $a + b$, and to divide their Sum by that Power, it will be convenient to write 1 and q for a and b; having taken $q : 1 :: b : a$: and to uſe a Table of *Logarithms*.

As in the Example of *Art.* 17th, where we had to compute $\frac{b^9 + 9ab^8 + 36a^2b^7}{\overline{a+b}|^9}$, a being $= 3$, $b = 5$; we ſhall have $q = \frac{5}{3}$, and, inſtead of the former, we are now to compute the quantity $\frac{q^9 + 9q^8 + 36q^7}{\overline{1+q}|^9} = \frac{q^7 \times \overline{q^2 + 9q + 36}}{\overline{1+q}|^9}$.

Now the Factor $q^2 + 9q + 36$ being $\frac{25}{9} + \frac{45}{3} + 36 = \frac{484}{9}$,
Whoſe Logarithm is $L.\ \underline{484 - L.\ 9} =$ - - 1.7306029
Add the Log. of q^7, or $7 \times \overline{L.\ 5 - L.\ 3} =$ - - 1.5529409
And from the Sum - - - - - - 3.2835438
Subtract the Log. of $\overline{1+q}|^9$, or $9 \times \overline{L.\ 8 - L.\ 3} =$ 3.8337183
So ſhall the Remainder - - - - — 1.4498255
be the Logarithm of *B*'s chance, *viz.* 0.281725
And the Complement of this to Unity 0.718275 is the Chance of *A*, in that Problem of *Art.* 17th.

An Operation of this kind will ſerve in moſt caſes that occur: but if the Power is very high, and the number of terms to be ſummed exceſſively great, we muſt have recourſe to other Rules; which ſhall be given hereafter.

II. When the Ratio of Chances, which we ſhall call that of R to S, comes out in larger numbers than we have occaſion for; it may be reduced to its *leaſt exacteſt* Terms, in the Method propoſed by Dr. *Wallis*, *Huygens*, and others. As thus;

Divide the greater Term R by the leſſer S; the laſt Diviſor by the Remainder; and ſo on continually, as in finding a common Diviſor: and let the ſeveral Quotients, in the order they ariſe, be repreſented by the Letters *a*, *b*, *c*, *d*, *e*, &c. Then the Ratio $\frac{S}{R}$, of the leſſer Term to the greater, will be contained in this fractional Series.

$$\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\&c.}}}}}$$

whoſe Terms, from the beginning, being reduced to one Fraction, will perpetually approach to the juſt Value of the Ratio $\frac{S}{R}$; differing from it in *exceſs* and in *defect*, alternately: ſo that if you ſtop at a Denominator that ſtands in the 1ſt, 3d, 5th, *&c.* place; as at *a*, *c*, *e*, &c. the Reſult of the Terms will exceed the juſt Value of the Ratio $\frac{S}{R}$; but if you ſtop at an even place, as at *b*, *d*, *f*, &c. it will fall ſhort of it.

EXAMPLE I.

If it is required to reduce the Ratio juſt now found $\frac{281725}{718275}$, or $\frac{11269}{28731}$ $\left(=\frac{S}{R}\right)$ to lower Terms; and which ſhall exhibit its juſt Quantity the neareſt that is poſſible in Terms ſo low: The Quotients, found as above, will be; $a=2$, $b=1$, $c=1$, $d=4$, $e=1$, $f=1$, $g=5$. And,

1°. The firſt Term $\frac{1}{a}$, or $\frac{1}{2}$, gives the Ratio too great; becauſe its conſequent *a* is too little.

2°. The Reſult of the two firſt Terms $\cfrac{1}{a+\cfrac{1}{b}}=\cfrac{1}{2+\cfrac{1}{1}}=\frac{1}{3}$, is leſs than $\frac{S}{R}$, altho' it comes nearer it than $\frac{1}{2}$ did: becauſe $\frac{1}{b}=1$, which we added to the Denominator 2, exceeds its juſt Quantity $\cfrac{1}{b+\cfrac{1}{c+\&c.}}$

3°. The three firſt Terms $\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c}}}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1}}}$; which reduced are $\cfrac{1}{2+\cfrac{1}{2}}=\frac{2}{5}$ exceeds the Ratio $\frac{S}{R}$: becauſe what we added to the Denominator b exceeding its juſt Quantity $\frac{1}{1+\frac{1}{4\,\&c.}}$ makes $\frac{1}{b+\frac{1}{c}}$ too little, and conſequently the whole Fraction too great.

In the ſame manner, the following Approximation $\frac{1}{a+\frac{1}{b+\frac{1}{c+\frac{1}{d}}}} = \frac{1}{2+\frac{1}{1+\frac{1}{1+\frac{1}{4}}}} = \frac{9}{23}$, tho' juſter than the preceding, errs a little in defect. And ſo of the reſt.

But to ſave unneceſſary trouble; and to prevent any miſtake either in the Operation itſelf, or in diſtinguiſhing the Ratios that exceed or fall ſhort of their juſt Quantity; we may uſe Mr. *Cotes*'s Rule; which is to this purpoſe.

Write $S : R$ at the head of two Columns, under the Titles *greater*, and *leſs*. And place under them the two firſt *Ratios* that are found; as in our Example $1 : 2$, and $1 : 3$. Multiply the Terms of this laſt Ratio by the *third* Denominator c, and write the Products under the Terms of the firſt Ratio $1 : 2$. So ſhall the Sums of the Antecedents and Conſequents give a juſter Ratio $2 : 5$, belonging to the left-hand Column. Multiply the terms of this laſt by the 4th Quotient d $(= 4)$, and the Products added to $1 : 3$ give the Ratio $9 : 23$, belonging to the right-hand Column. This laſt multiplied by e $(= 1)$, and the Products transferred to the left hand Column, and added to the Ratio that ſtood laſt there, give the Ratio $11 : 28$. And ſo of the reſt, as in the Scheme below.

$$\begin{array}{rl|ll}
 & \textit{greater} & \textit{leſs} & \\
 & S : R & S : R & \\
\hline
\frac{1}{a} = & 1 : 2 & 1 : 3 = \frac{1}{a+\frac{1}{b}} & \\
 & \underline{1 : 3} & \times\ c = 1 & \\
 & 2 : 5 \times d = 4 & \underline{8 : 20} & \\
 & \underline{9 : 23} & 9 : 23 \times e = 1 & \\
 & 11 : 28 \times f = 1 & \underline{11 : 28} & \\
 & \underline{100 : 255} & 20 : 51 \times g = 5 & \\
 & 111 : 283 \quad \&c. & &
\end{array}$$

This Method is particularly uſeful, when ſurd numbers, which have no Termination at all, enter into any Solution.

EXAMPLE II.

It will be found in the Reſolution of our firſt Problem that the proportion of Chances there inquired into $\left(\frac{R}{S}\right)$ is that of 1 to $\sqrt[3]{2} - 1$, or of 1 to 0.259921 &c. Whence our Quotients will be; $a = 3$, $b = 1$, $c = 5$, $d = 1$, $e = 1$, $f = 4$, &c.

And the Operation will ſtand as below.

$$
\begin{array}{rcccl}
 & \textit{greater} & & \textit{leſs} & \\
 & S : R & & S : R & \\
\hline\hline
\frac{1}{a} = & 1 : 3 & & 1 : 4 & = \frac{1}{a+\frac{1}{b}} \\
 & \underline{5 : 20} & & \times\ c = 5 & \\
 & 6 : 23 & \times\ d = 1 & \underline{6 : 23} & \\
 & & & 7 : 27 & \\
 & \underline{7 : 27} & & \times\ e = 1 & \\
 & 13 : 50 & & & \\
 & \times\ f = 4 & & \underline{52 : 200} & \\
 & & & 59 : 227 & \\
 & & \&c. & &
\end{array}
$$

End of the Introduction.

Solutions of ſeveral ſorts of Problems, deduced from the Rules laid down in the Introduction.

PROBLEM I.

If A *and* B *play with ſingle Bowls, and ſuch be the ſkill of* A *that he knows by Experience he can give* B *two Games out of three; what is the proportion of their ſkill, or what are the Odds, that* A *may get any one Game aſſigned?*

Solution.

LET the proportion of Odds be as z to 1; now ſince *A* can give *B* 2 Games out of 3, *A* therefore may upon an equality of Play undertake to win 3 Games together: but the probability of his winning the firſt time is $\frac{z}{z+1}$, and, by *the 8th Article of the Introduction*, the probability of his winning three times together is $\frac{z}{z+1} \times \frac{z}{z+1} \times \frac{z}{z+1}$ or $\frac{z^3}{\overline{z+1}|^3}$. Again, becauſe *A* and *B* are ſuppoſed to play upon equal terms, the probability which *A* has of winning three times together ought to be expreſſed by $\frac{1}{2}$; we have therefore the Equation $\frac{z^3}{\overline{z+1}|^3} = \frac{1}{2}$, or $2z^3 = \overline{z+1}|^3$, and extracting the cube-root on both ſides, $z\sqrt[3]{2} = z + 1$; wherefore $z = \frac{1}{\sqrt[3]{2} - 1}$, and conſequently the Odds that *A* may get any one Game aſſigned are as $\frac{1}{\sqrt[3]{2} - 1}$ to 1, or as 1 to $\sqrt[3]{2} - 1$, that is in this caſe as 50 to 13 very near.

COROLLARY.

By the ſame proceſs of inveſtigation as that which has been uſed in this Problem, it will be found that if *A* can, upon an equality of Chance, undertake to win n times together, then he may juſtly lay the Odds of 1 to $\sqrt[n]{2} - 1$, that he wins any one Game aſſigned.

PROBLEM II.

If A *can without advantage or diſadvantage give B* 1 *Game out of* 3; *what are the Odds that* A *ſhall take any one Game aſſigned? Or in other terms, what is the proportion of the Chances they reſpectively have of winning any one Game aſſigned? Or what is the proportion of their ſkill?*

SOLUTION.

Let the proportion be as z to 1: and ſince *A* can give *B* 1 Game out of 3; therefore *A* can upon an equality of play undertake to win 3 Games before *B* gets 2: now it appears, *by the* 17*th Art. of the Introduction*, that in this caſe the Binomial $z + 1$ ought to be raiſed to its fourth power, which will be $z^4 + 4z^3 + 6zz + 4z + 1$; and that the Expectation of the firſt will be to the Expectation of the ſecond, as the two firſt terms to the three laſt: but theſe Expectations are equal by hypotheſis, therefore $z^4 + 4z^3 = 6zz + 4z + 1$: which Equation being ſolved, z will be found to be 1.6 very near; wherefore the proportion required will be as 1.6 to 1, or 8 to 5.

PROBLEM III.

To find in how many Trials an Event will probably happen, or how many Trials will be neceſſary to make it indifferent to lay on its Happening or Failing; ſuppoſing that a *is the number of Chances for its happening in any one Trial, and* b *the number of Chances for its failing.*

SOLUTION.

Let x be the number of Trials; then *by the* 16*th Art. of the Introd.* b^x will repreſent the number of Chances for the Event to fail x times ſucceſſively, and $\overline{a+b}|^x$ the whole number of Chances for happening or failing, and therefore $\frac{b^x}{\overline{a+b}|^x}$ repreſents the probability of the Event's failing x times together: but by ſuppoſition that Probability is equal to the probability of its happening once at leaſt in that number of Trials; wherefore either of thoſe two Probabilities may be expreſſed by the fraction $\frac{1}{2}$: we have therefore the Equation $\frac{b^x}{\overline{a+b}|^x} = \frac{1}{2}$, or $\overline{a+b}|^x = 2b^x$, from whence is deduced the Equation $x \log. \overline{a+b} = x \log. b + \log. 2$; and therefore $x = \frac{\text{Log. } 2}{\text{Log. } \overline{a+b} - \log. b}$.

Moreover, let us reaſſume the Equation $\overline{a+b}|^x = 2b^x$, wherein let us ſuppoſe that $a, b :: 1, q$; hence the ſaid Equation will be changed into this $\overline{1+\frac{1}{q}}|^x = 2$. Or $x \times \log. \overline{1+\frac{1}{q}} = \log. 2$. In this Equation, if q be equal to 1, x will likewiſe be equal to 1; but if q differs from Unity, let us in the room of log. $\overline{1+\frac{1}{q}}$ write its value expreſſed in a Series; *viz.*

$$\frac{1}{q} - \frac{1}{2qq} + \frac{1}{3q^3} - \frac{1}{4q^4} + \frac{1}{5q^5} - \frac{1}{6q^6}, \text{ \&c.}$$

We have therefore the Equation $\frac{x}{q} - \frac{x}{2qq}$, &c. $= \log. 2$. Let us now ſuppoſe that q is infinite, or pretty large in reſpect to *Unity*, and then the firſt term of the Series will be ſufficient; we ſhall therefore have the Equation $\frac{x}{q} = \log. 2$, or $x = q \log. 2$. But it is to be obſerved in this place that the Hyperbolic, not the Tabular, Logarithm of 2, ought to be taken, which being 0.693, &c. or 0.7 nearly, it follows that $x = 0.7q$ nearly.

Thus we have aſſigned the very narrow limits within which the ratio of x to q is comprehended; for it begins with unity, and terminates at laſt in the ratio of 7 to 10 very near.

But x ſoon converges to the limit $0.7q$, ſo that this value of x may be aſſumed in all caſes, let the value of q be what it will.

Some uſes of this Problem will appear by the following Examples.

EXAMPLE I.

Let it be propoſed to find in how many throws one may undertake with an equality of Chance, to throw two Aces with two Dice.

The number of Chances upon two Dice being 36, out of which there is but one chance for two Aces, it follows that the number of

Chances againſt it is 35; multiply therefore 35 by 0.7, and the product 24.5 will ſhew that the number of throws requiſite to that effect will be between 24 and 25.

EXAMPLE 2.

To find in how many throws of three Dice, one may undertake to throw three Aces.

The number of all the Chances upon three Dice being 216, out of which there is but one Chance for 3 Aces, and 215 againſt it, it follows that 215 ought to be multiplied by 0.7; which being done, the product 150.5 will ſhew that the number of Throws requiſite to that effect will be 150, or very near it.

EXAMPLE 3.

In a Lottery whereof the number of blanks is to the number of prizes as 39 *to* 1, (*ſuch as was the Lottery in* 1710) *to find how many Tickets one muſt take to make it an equal Chance for one or more Prizes.*

Multiply 39 by 0.7, and the product 27.3 will ſhew that the number of Tickets requiſite to that effect will be 27 or 28 at moſt.

Likewiſe in a Lottery whereof the number of Blanks is to the number of Prizes as 5 to 1, multiply 5 by 0.7, and the product 3.5 will ſhew that there is more than an equality of Chance in 4 Tickets for one or more Prizes, but leſs than an equality in three.

REMARK.

In a Lottery whereof the Blanks are to the Prizes as 39 to 1, if the number of Tickets in all were but 40, the proportion abovementioned would be altered, for 20 Tickets would be a ſufficient number for the juſt Expectation of the ſingle Prize; it being evident that the Prize may be as well among the Tickets which are taken, as among thoſe that are left behind.

Again if the number of Tickets in all were 80, ſtill preſerving the proportion of 39 Blanks to one Prize, and conſequently ſuppoſing 78 Blanks to 2 Prizes, this proportion would ſtill be altered; for by the Doctrine of Combinations, whereof we are to treat afterwards, it will appear that the Probability of taking one Prize or both in 20 Tickets would be but $\frac{139}{316}$, and the Probability of taking none would be $\frac{177}{316}$; wherefore the Odds againſt taking any Prize would be as 177 to 139, or very near as 9 to 7.

And by the ſame Doctrine of Combinations, it will be found that 23 Tickets would not be quite ſufficient for the Expectation of a Prize in this Lottery; but that 24 would rather be too many: ſo that one might with advantage lay an even Wager of taking a Prize in 24 Tickets.

If the proportion of 39 to 1 be oftner repeated, the number of Tickets requiſite for the equal Chance of a Prize, will ſtill increaſe with that repetition; yet let the proportion of 39 to 1 be repeated never ſo many times, nay an infinite number of times, the number of Tickets requiſite for the equal Chance of a Prize would never exceed $\frac{7}{10}$ of 39, that is about 27 or 28.

Wherefore if the proportion of the Blanks to the Prizes is often repeated, as it uſually is in Lotteries; the number of Tickets requiſite for a Prize will always be found by taking $\frac{7}{10}$ of the proportion of the Blanks to the Prizes.

Now in order to have a greater variety of Examples to try this Rule by, I have thought fit here to annex a *Lemma* by me publiſhed for the firſt time in the year 1711, and of which the inveſtigation for particular reaſons was deferred till I gave it in my *Miſcellanea Analytica* anno 1731.

LEMMA.

To find how many Chances there are upon any number of Dice, each of them of the ſame number of Faces, to throw any given number of points.

SOLUTION.

Let $p+1$ be the number of points given, n the number of Dice, f the number of Faces in each Die: make $p-f=q$, $q-f=r$, $r-f=s$, $s-f=t$, &c. and the number of Chances required will be

$$\begin{array}{ll}
+\frac{p}{1}\times\frac{p-1}{2}\times\frac{p-2}{3}, \&c. & \\
-\frac{q}{1}\times\frac{q-1}{2}\times\frac{q-2}{3}, \&c. & \times\frac{n}{1} \\
+\frac{r}{1}\times\frac{r-1}{2}\times\frac{r-2}{3}, \&c. & \times\frac{n}{1}\times\frac{n-1}{2} \\
-\frac{s}{1}\times\frac{s-1}{2}\times\frac{s-2}{3}, \&c. & \times\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3} \\
+\&c. &
\end{array}$$

Which Series's ought to be continued till ſome of the Factors in each product become either $=0$, or negative.

N. B. So many Factors are to be taken in each of the products $\frac{p}{1}\times\frac{p-1}{2}\times\frac{p-2}{3}$, &c. $\frac{q}{1}\times\frac{q-1}{2}\times\frac{q-2}{3}$, &c. as there are Units in $n-1$.

Thus for Example, let it be required to find how many Chances there are for throwing 16 Points with four Dice; then making $p+1=16$, we have $p=15$, from whence the number of Chances required will be found to be

$$\begin{array}{ll} +\frac{15}{1}\times\frac{14}{2}\times\frac{13}{3} & =+455 \\ -\frac{9}{1}\times\frac{8}{2}\times\frac{7}{3}\times\frac{4}{1} & =-336 \\ +\frac{3}{1}\times\frac{2}{2}\times\frac{1}{3}\times\frac{4}{1}\times\frac{3}{2} & =+\ 6 \end{array}$$

But $455-336+6=125$, and therefore one hundred and twenty-five is the number of Chances required.

Again, let it be required to find the number of Chances for throwing ſeven and twenty Points with ſix Dice; the operation will be

$$\begin{array}{ll} +\frac{26}{1}\times\frac{25}{2}\times\frac{24}{3}\times\frac{23}{4}\times\frac{22}{5} & =+65780 \\ -\frac{20}{1}\times\frac{19}{2}\times\frac{18}{3}\times\frac{17}{4}\times\frac{16}{5}\times\frac{6}{1} & =-93024 \\ +\frac{14}{1}\times\frac{13}{2}\times\frac{12}{3}\times\frac{11}{4}\times\frac{10}{5}\times\frac{6}{1}\times\frac{5}{2} & =+30030 \\ -\frac{8}{1}\times\frac{7}{2}\times\frac{6}{3}\times\frac{5}{4}\times\frac{4}{5}\times\frac{6}{1}\times\frac{5}{2}\times\frac{4}{3} & =-\ 1120 \end{array}$$

Wherefore $65780-93024+30030-1120=1666$ is the number of Chances required.

Let it be farther required to aſſign the number of Chances for throwing fifteen Points with ſix Dice.

$$\begin{array}{ll} +\frac{14}{1}\times\frac{13}{2}\times\frac{12}{3}\times\frac{11}{4}\times\frac{10}{5} & =+2002 \\ -\frac{8}{1}\times\frac{7}{2}\times\frac{6}{3}\times\frac{5}{4}\times\frac{4}{5}\times\frac{6}{1} & =-\ 336 \end{array}$$

But $2002-336=1666$ which is the number required.

COROLLARY.

All the points equally diſtant from the Extremes, that is from the leaſt and greateſt number of Points that are upon the Dice, have the ſame number of Chances by which they may be produced; wherefore if the number of points given be nearer to the greater Extreme than to the leſſer, let the number of points given be ſubtracted from the Sum of the Extremes, and work with the remainder; by which means the Operation will be ſhortened.

Thus if it be required to find the number of Chances for throwing 27 Points with 6 Dice: let 27 be ſubtracted from 42, Sum of the Extremes 6 and 36, and the remainder being 15, it may be concluded that the number of Chances for throwing 27 Points is the ſame as for throwing 15 Points.

Although, as I have ſaid before, the Demonſtration of this Lemma may be had from my *Miſcellanea*; yet I have thought fit, at the deſire of ſome Friends, to transfer it to this place.

DEMONSTRATION.

1°. Let us imagine a Die ſo conſtituted as that there ſhall be upon it one ſingle Face marked 1, then as many Faces marked 11 as there are Units in r, and as many Faces marked 111 as there are Units in rr, and ſo on; that the geometric Progreſſion $1 + r + rr + r^3 + r^4 + r^5 + r^6 + r^7 + r^8$, &c. continued to ſo many Terms as there are different Denominations in the Die, may repreſent all the Chances of one Die: this being ſuppoſed, it is very plain that in order to have all the Chances of two ſuch Dice, this Progreſſion ought to be raiſed to its Square, and that to have all the Chances of three Dice, the ſame Progreſſion ought to be raiſed to its Cube; and univerſally, that if the number of Dice be expreſſed by n, that Progreſſion ought accordingly to be raiſed to the Power n. Now ſuppoſe the number of Faces in each Die to be f, then the Sum of that Progreſſion will be $\frac{1-r^f}{1-r}$; and conſequently every Chance that can happen upon n Dice, will be expreſſed by ſome Term of the Series that reſults from the Fraction $\frac{1-r^f}{1-r}$ raiſed to the power n. But as the leaſt number of Points, that can be thrown with n Dice, is n Units, and the next greater $n+1$, and the next $n+2$, &c. it is plain that the firſt Term of the Series will repreſent the number of Chances for throwing n Points, and that the ſecond Term of the Series will repreſent the number of Chances for throwing $n+1$ Points, and ſo on. And that therefore if the number of Points to be thrown be expreſſed by $p+1$, it will be but aſſigning that Term in the Series of which the diſtance from the firſt ſhall be expreſſed by $p+1-n$.

But the Series which would reſult from the raiſing of the Fraction $\frac{1-r^f}{1-r}$ to the Power n, is the Product of two other Series, whereof one is $1 + nr + \frac{n}{1} \times \frac{n+1}{2} rr + \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} r^3$, &c. the other is $1 - nr^f + \frac{n}{1} \times \frac{n-1}{2} r^{2f} - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} r^{3f} +$ &c. Wherefore, if theſe two Series be multiplied together, all the Terms of the product will ſeverally anſwer the ſeveral numbers of Chances that are upon n Dice.

And therefore if the number of Points to be thrown be expreſſed by $p+1$, it is but collecting all the Terms which are affected by the Power r^{p+1-n}, and the Sum of thoſe Terms will anſwer the Queſtion propoſed.

But in order to find readily all the Terms which are affected by the Power r^{p+1-n}, let us ſuppoſe, for ſhortneſs ſake, $p+1-n=l$; and let us ſuppoſe farther that Er^l is that Term, in the firſt Series, of which the diſtance from its firſt Term is l; let alſo Dr^{l-f} be that Term, in the firſt Series, of which the diſtance from its firſt Term is $l-f$, and likewiſe let Cr^{l-2f} be that Term, in the firſt Series, of which the diſtance from its firſt Term is denoted by $l-2f$, and ſo on, making perpetually a regreſs towards the firſt Term. This being laid down, let us write all thoſe Terms in order, thus

$$Er^l + Dr^{l-f} + Cr^{l-2f} + Br^{l-3f}, \text{ \&c.}$$

and write underneath the Terms of the ſecond Series, in their natural order. Thus

$$\begin{array}{llll} Er^l & + Dr^{l-f} & + Cr^{l-2f} & + Br^{l-3f}, \text{ \&c.} \\ 1 & - nr^f & + \frac{n}{1} \times \frac{n-1}{2} r^{2f} & + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} r^{3f}, \text{ \&c.} \end{array}$$

then multiplying each Term of the firſt Series by each correſponding Term of the ſecond, all the Terms of the product, *viz.*

$$Er^l - nDr^l + \frac{n}{1} \times \frac{n-1}{2} Cr^l - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} Br^l, \text{ \&c.}$$

will be affected with the ſame power r^l.

Now the Coefficient E containing ſo many factors $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$, &c. as there are Units in l; it is plain, that when the Denominators of thoſe factors are continued beyond a certain number of them, denominated by $n-1$, then the following Denominators will be n, $n+1$, $n+2$, &c. which being the ſame as the firſt Terms of the Numerators, it follows that if from the value of the Coefficient E be rejected thoſe Numerators and Denominators which are equal, there will remain out of the Numerators, written in an inverted order, the Terms $n+l-1$, $n+l-2$, $n+l-3$, &c. of which the laſt will be $l+1$; and that, out of the Denominators written in their natural order, there will remain 1, 2, 3, 4, 5, &c. of which the laſt will be $n-1$: all which things depend intirely on the nature of an Arithmetic Progreſſion. Wherefore the firſt Term

Er^l is $= \frac{n+l-1}{1} \times \frac{n+l-2}{2} \times \frac{n+l-3}{3} \cdots\cdots\cdots \frac{l+1}{n-1} r^l$. Now in the room of l, ſubſtitute its value $p+1-n$, then $Er^l = \frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3}$, &c. $\times\, r^l$, and in the ſame manner will the ſecond Term

$-nDr^l$ be $= -\frac{p-f}{1} \times \frac{p-f-1}{2} \times \frac{p-f-2}{3}$ &c. $\times\, nr^l$, and alſo the third Term

$+\frac{n}{1} \times \frac{n-1}{2} Cr^l$ will be $= +\frac{p-2f}{1} \times \frac{p-2f-1}{2} \times \frac{p-3f-2}{3}$ &c. $\times \frac{n}{1} \times \frac{n-1}{2} r^l$, and ſo on. Suppoſe now $r=1$, $p-f=q$, $q-f=r$, $r-f=s$, &c. and you ſhall have the very Rule given in our Lemma.

Now to add one Example more to our third Problem, let it be required to find in how many throws of 6 Dice one may undertake to throw 15 Points preciſely.

The number of Chances for throwing 15 Points being 1666, and the whole number of Chances upon 6 Dice being 46656, it follows that the number of Chances for failing is 44990; wherefore dividing 44990 by 1666, and the quotient being 27 nearly, multiply 27 by 0.7, and the product 18.9 will ſhew that the number of throws requiſite to that effect will be very near 19.

PROBLEM IV.

To find how many Trials are neceſſary to make it equally probable that an Event will happen twice, ſuppoſing that a *is the number of Chances for its happening in any one Trial, and* b *the number of Chances for its failing.*

SOLUTION.

Let x be the number of Trials: then from what has been demonſtrated *in the* 16th *Art. of the Introd.* it follows that $b^x + xab^{x-1}$ is the number of Chances whereby the Event may fail, $\overline{a+b}|^{x}$ comprehending the whole number of Chances whereby it may either happen or fail, and conſequently the probability of its failing is $\frac{b^{x}+xab^{x-1}}{\overline{a+b}|^{x}}$: but, by Hypotheſis, the Probabilities of happening and failing are equal; we have therefore the Equation $\frac{b^{x}+xab^{x-1}}{\overline{a+b}|^{x}} = \frac{1}{2}$, or $\overline{a+b}|^{x} = 2b^{x} + 2xab^{x-1}$, or making $a, b :: 1, q$, $\overline{1+\frac{1}{q}}|^{x} = 2 + \frac{2x}{q}$. Now if in this Equation we ſuppoſe $q = 1$, x will be found $= 3$, and if we ſuppoſe q infinite, and alſo $\frac{x}{q} = z$, we ſhall have the Equation $z = \log. 2 + \log. \overline{1+z}$, in which taking the value of z, either by Trial or otherwiſe, it will be found $= 1.678$ nearly; and therefore the value of x will always be between the limits $3q$ and $1.678q$, but will ſoon converge to the laſt of theſe limits; for which reaſon, if q be not very ſmall, x may in all caſes be ſuppoſed $= 1.678q$; yet if there be any ſuſpicion that the value of x thus taken is too little, ſubſtitute this value in the original Equation $\overline{1+\frac{1}{q}}|^{x} = 2 + \frac{2x}{q}$, and note the Error. Then if it be worth taking notice of, increaſe a little the value of x, and ſubſtitute again this new value of x in the aforeſaid Equation; and noting the new Error, the value of x may be ſufficiently corrected by applying the Rule which the Arithmeticians call double falſe Poſition.

EXAMPLE 1.

To find in how many throws of three Dice one may undertake to throw three Aces twice.

The number of all the Chances upon three Dice being 216, out of which there is but 1 Chance for three Aces, and 215, againſt it; multiply 215 by 1.678 and the product 360.8 will ſhew that the number of throws requiſite to that effect will be 361, or very near it.

EXAMPLE 2.

To find in how many throws of 6 Dice one may undertake to throw 15 Points twice.

The number of Chances for throwing 15 Points is 1666, the number of Chances for miſſing 44990; let 44990 be divided by

1666, the Quotient will be 27 very near: wherefore the Chances for throwing and miſſing 15 Points are as 1 to 27 reſpectively; multiply therefore 27 by 1.678, and the product 45.3 will ſhew that the number of Chances requiſite to that effect will be 45 nearly.

EXAMPLE 3.

In a Lottery whereof the number of Blanks is to the Number of Prizes as 39 to 1: to find how many Tickets muſt be taken to make it as probable that two or more benefits will be taken as not.

Multiply 39 by 1.678 and the product 65.4 will ſhew that no leſs than 65 Tickets will be requiſite to that effect.

PROBLEM V.

To find how many Trials are neceſſary to make it equally probable that an Event will happen three, four, five, &c. times; ſuppoſing that a *is the number of Chances for its happening in any one Trial, and* b *the number of Chances for its failing.*

SOLUTION.

Let x be the number of Trials requiſite, then ſuppoſing as before $a, b :: 1, q$, we ſhall have the Equation $\overline{1+\frac{1}{q}}\Big|^{x} = 2 \times \overline{1 + \frac{x}{q} + \frac{x}{1} \times \frac{x-1}{2qq}}$, in the caſe of the triple Event; or $\overline{1+\frac{1}{q}}\Big|^{x} = 2 \times \overline{1 + \frac{x}{q} + \frac{x}{1} \times \frac{x-1}{2qq} + \frac{x}{1} \times \frac{x-1}{2} \times \frac{x-2}{3q^3}}$ in the caſe of the quadruple Event: and the law of the continuation of theſe Equations is manifeſt. Now in the firſt Equation if q be ſuppoſed $= 1$, then will x be $= 5$; if q be ſuppoſed infinite or pretty large in reſpect to Unity, then the aforeſaid Equation, making $\frac{x}{q} = z$, will be changed into this, $z = \log. 2 + \log. \overline{1 + z + \frac{1}{2}zz}$; wherein z will be found nearly $= 2.675$, wherefore x will always be between $5q$ and $2.675q$.

Likewiſe in the ſecond Equation, if q be ſuppoſed $= 1$, then will x be $= 7q$; but if q be ſuppoſed infinite or pretty large in reſpect to Unity, then $z = \log. 2 + \log. \overline{1 + z + \frac{1}{2}zz + \frac{1}{9}z^3}$;

whence z will be found nearly $= 3\ 6719$, wherefore x will be between $7q$ and $3.6719q$.

A TABLE *of the Limits.*

The Value of x will always be

For a ſingle Event, between	$1q$	and	$0.693q$
For a double Event, between	$3q$	and	$1.678q$
For a triple Event, between	$5q$	and	$2.675q$
For a quadruple Event, between	$7q$	and	$3.672q$
For a quintuple Event, between	$9q$	and	$4.670q$
For a ſextuple Event, between	$11q$	and	$5.668q$
&c.			

And if the number of Events contended for, as well as the number q be pretty large in reſpect to Unity; the number of Trials requiſite for thoſe Events to happen n times will be $\frac{2n-1}{2}q$, or barely nq.

REMARK.

From what has been ſaid we may plainly perceive that altho' we may, with an equality of Chance, contend about the happening of an Event once in a certain number of Trials, yet we cannot, without diſadvantage, contend for its happening twice in double that number of Trials, or three times in triple that number, and ſo on. Thus, altho' it be an equal Chance, or rather more than an Equality, that I throw two Aces with two Dice in 25 throws, yet I cannot undertake that the two Aces ſhall come up twice in 50 throws, the number requiſite for it being 58 or 59; much leſs can I undertake that they ſhall come up three times in 75 throws, the number requiſite for it being between 93 and 94: ſo that the Odds againſt the happening of two Aces in the firſt throw being 35 to 1, I cannot undertake that in a very great number of Trials, the happening ſhall be oftner than in the proportion of 1 to 35. And therefore we may lay down this general Maxim, that Events at long run will not happen oftner than in the proportion of the Chances they have to happen in any one Trial; and that if we aſſign any other proportion varying never ſo little from that, the Odds againſt us will increaſe continually.

To this may be objected, that from the premiſes it would ſeem to follow, that if two equal Gameſters were to play together for a conſiderable time, they would part without Gain or Loſs on either ſide: but the anſwer is eaſy; the longer they play the greater Probability there is of an increaſe of abſolute Gain or Loſs; but at the ſame time, the greater Probability there is alſo of a decreaſe, in reſpect to the number of Games played. Thus if 100 Games produce a difference of 4 in the winnings or loſings, and 200 Games produce a difference of 6, there will be a greater proportion of Equality in the ſecond caſe than in the firſt.

PROBLEM VI.

Three Gameſters A, B, C *play together on this condition, that he ſhall win the Set who has ſooneſt got a certain number of Games; the proportion of the Chances which each of them has to get any one Game aſſigned, or which is the ſame thing, the proportion of their ſkill, being reſpectively as* a, b, c. *Now after they have played ſome time, they find themſelves in this circumſtance, that* A *wants* 1 *Game of being up,* B 2 *Games, and* C 3 *Games; the whole Stake amongſt them being ſuppoſed* 1; *what is the Expectation of each?*

SOLUTION. I.

In the circumſtance the Gameſters are in, the Set will be ended in 4 Games at moſt; let therefore $a+b+c$ be raiſed to the fourth power, which will be $a^4+4a^3b+6aabb+4ab^3+b^4+4a^3c+12aabc+4b^3c+6aacc+12abcc+6bbcc+4ac^3+12acbb+4bc^3+c^4$.

The terms $a^4+4a^3b+4a^3c+6aacc+12aabc+12abcc$, wherein the dimenſions of a are equal to or greater than the number of Games which A wants, wherein alſo the Dimenſions of b and c are leſs than the number of Games which B and C reſpectively want, are intirely favourable to A, and are part of the Numerator of his Expectation.

In the ſame manner, the terms $b^4+4b^3c+6bbcc$ are intirely favourable to B.

And likewiſe the terms $4bc^3+c^4$ are intirely favourable to C.

The reſt of the terms are common, as favouring partly one of the Gameſters, partly one or both of the other; wherefore theſe Terms are ſo to be divided into their parts, that the parts, reſpectively favouring each Gameſter, may be added to his Expectation.

Take therefore all the terms which are common, *viz.* $6aabb$, $4ab^3$, $12abcc$, $4ac^3$, and divide them actually into their parts; that is, 1°, $6aabb$ into $aabb$, $abab$, $abba$, $baab$, $baba$, $bbaa$. Out of theſe ſix parts, one part only, *viz.* $bbaa$ will be found to favour B, for 'tis only in this term that two Dimenſions of b are placed before one ſingle Dimenſion of a, and therefore the other five parts belong to A; let therefore $5aabb$ be added to the Expectation of A, and $1aabb$ to the Expectation of B. 2°. Divide $4ab^3$, into its parts $abbb$, $babb$, $bbab$, $bbba$; of theſe parts there are two belonging to A, and the other two to B; let therefore $2ab^3$ be added to the expectation of each. 3°. Divide $12abbc$ into its parts; and eight of them will belong to A, and 4 to B; let therefore $8abbc$ be added to the Expectation of A, and $4abbc$ to the Expectation of B. 4°. Divide $4ac^3$ into its parts, three of which will be found to be favourable to A, and one to C; add therefore $3ac^3$ to the Expectation of A, and $1ac^3$ to the Expectation of C. Hence the Numerators of the ſeveral Expectations of A, B, C, will be reſpectively,

1. $a^4 + 4a^3b + 4a^3c + 6aacc + 12aabc + 12abcc + 5aabb + 2ab^3 + 8abbc + 3ac^3$.
2. $b^4 + 4b^3c + 6bbcc + 1aabb + 2ab^3 + 4abcc$.
3. $4bc^3 + 1c^4 + 1ac^3$.

The common Denominator of all their Expectations being $\overline{a+b+c}|^4$.

Wherefore if a, b, c, are in a proportion of equality, the Odds of winning will be reſpectively as 57, 18, 6, or as 19, 6, 2.

If n be the number of all the Games that are wanting, p the number of Gameſters, and a, b, c, d, &c. the proportion of the Chances which each Gameſter has reſpectively to win any one Game aſſigned; let $a + b + c + d$, &c. be raiſed to the power $n + 1 - p$, and then proceed as before.

REMARK.

This is one general Method of Solution. But the ſimpler and more common Caſes may be managed with very little trouble. As,

1°. Let A and B want one game each, and C two games. Then the following game will either put him in the ſame ſituation as A and B, entitling him to $\frac{1}{3}$ of the Stake; of which there is 1 Chance: or will give the whole Stake to A or B; and of this there are two Chances. C's Expectation therefore is worth $\dfrac{1 \times \frac{1}{3} + 2 \times 0}{3}$

(*Introd. Art.* 5.) $=\frac{1}{9}$. Take this from the Stake 1, and the Remainder $\frac{8}{9}$, to be divided equally between *A* and *B*, makes the expectations of *A*, *B*, *C*, to be 4, 4, 1, reſpectively; to the common Denominator 9.

2°. Let *A* want 1 Game, *B* and *C* two games each. Then the next Game will either give *A* the whole Stake; or, one of his Adverſaries winning, will reduce him to the Expectation $\frac{4}{9}$, of the former Caſe. His preſent Expectation therefore is $\frac{1\times1+2\times\frac{4}{9}}{3}=\frac{17}{27}$: and the Complement of this to Unity, *viz.* $\frac{10}{27}$, divided equally between *B* and *C*, gives the three Expectations, 17, 5, 5, the common Denominator being 27.

3°. *A* and *B* wanting each a Game, let *C* want 3. In this Caſe, *C* has 2 Chances for 0, and 1 Chance for the Expectation $\frac{1}{9}$, of *Caſe* 1. That is, his Expectation is $\frac{1}{27}$; and thoſe of *A* and *B* are $\frac{13}{27}$, each.

4°. Let the Games wanting to *A*, *B*, and *C*, be 1, 2, 3, reſpectively: then *A* winning gets the Stake 1; *B* winning, *A* is in *Caſe* 3, with the Expectation $\frac{13}{27}$, or *C* winning, he has, as in *Caſe* 2, the Expectation $\frac{17}{27}$. Whence his preſent Expectation is $\frac{1}{3}\times\overline{1+\frac{13}{27}+\frac{17}{27}}=\frac{57}{81}$.

Again, *A* winning, *B* gets 0; himſelf winning, he acquires (*Caſe* 3.) the Expectation $\frac{13}{27}$. And, *C* winning, he is in *Caſe* 2, with the Expectation $\frac{5}{27}$. His preſent Expectation therefore is $\frac{1}{3}\times\overline{0+\frac{13}{27}+\frac{5}{27}}=\frac{18}{81}$. Add this to the Expectation of *A*, which was $\frac{57}{81}$; the Sum is $\frac{75}{81}$: and the Complement of this to Unity, which is $\frac{6}{81}$, is the Expectation of C.

Or to find *C*'s Expectation directly: *A* winning, *C* has 0; *B* winning, he has the Expectation $\frac{1}{27}$, (*Caſe* 3.) and, himſelf winning, he has $\frac{5}{27}$, as in *Caſe* 2: In all, $\frac{1}{3}\times\overline{0+\frac{1}{27}+\frac{5}{27}}=\frac{6}{81}$.

And thus, aſcending gradually through all the inferior Caſes, or by the general Rule, we may compoſe a Table of *Odds* for 3 Gamſters, ſuppoſed of equal Skill; like that for 2 Gameſters in *Art.* 17th of the Introduction.

Table for 3 Gameſters.

Games wanting.			Odds.			Games wanting.			Odds.			Games wanting.			Odds.		
A.	*B.*	*C*	*a.*	*b.*	*c.*	*A*	*B.*	*C.*	*a.*	*b.*	*c*	*A.*	*B.*	*C.*	*a.*	*b.*	*c.*
1	1	2	4	4	1	1	2	3	19	6	2	2	2	4	338	338	53
1	1	3	13	13	1	1	2	4	178	58	7	2	2	5	353	353	23
1	1	4	40	40	1	1	2	5	542	179	8	2	3	3	133	55	55
1	1	5	121	121	1	1	3	4	616	82	31	2	3	4	451	195	83
1	2	2	17	5	5	1	3	5	629	87	13	2	3	5	1433	635	119
1	3	3	65	8	8	2	2	3	34	34	13	&c.			&c.		

SOLUTION II. and more *General.*

It having been objected to the foregoing Solution, that when there are ſeveral Gameſters, and the number of games wanting amongſt them is conſiderable; the Operation muſt be tedious; and that there may be ſome danger of miſtake, in ſeparating and collecting the ſeveral parts of their Expectations, from the Terms of the Multinomial: I invented this other Solution, which was publiſhed in the VIIth Book of my *Miſcellanea Analytica*, A. D. 1730.

The Skill of the Gameſters *A*, *B*, *C*, &c. is now ſuppoſed to be as a, b, c, &c. reſpectively: and the Games they want of the Set are p, q, r, &c. Then in order to find the Chance of a particular Gameſter, as of *A*, or his Right in the Stake 1, we may proceed as follows.

1°. Write down Unity.

2°. Write down in order all the Letters b, c, d, &c. which denote the Skill of the Gameſters, excepting only the Letter which belongs to the Gameſter whoſe Chance you are computing; as in our Example, the Letter a is omitted.

3°. Combine the ſame Letters b, c, d, &c. by two's, three's, four's, &c.

4°. Of theſe Combinations, leave out or cancel all ſuch as make any Gameſter beſides *A*, the winner of the Set; that is, which give to *B*, q Games; to *C*, r Games, to *D*, s Games, &c.

5°. Multiply the whole by a^{p-1}.

6°. Prefix to each Product the Number of its *Permutations*, that is, of the different ways in which its Letters can be written *.

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* Of *Combinations* and *Permutations*, See Prob. xiv. & *ſeqq.*

7°. Let all the Products that are of the ſame dimenſion, that is, which contain the ſame number of Letters, be collected into different ſums.

8°. Let theſe ſeveral Sums, from the loweſt dimenſion upwards, be divided by the Terms of this Series,

$ſ^{p-1}$, $ſ^{p}$, $ſ^{p+1}$, $ſ^{p+2}$, &c. reſpectively: in which *Series* $ſ=a+b+c+d+$ &c.

9°. Laſtly, multiply the Sum of the Quotients by $\frac{a}{ſ}$, and the Product ſhall be the Chance or Expectation required; namely the Right of *A* in the Stake 1. And in the ſame way, the Expectations of the other Gameſters may be computed.

EXAMPLE.

Suppoſing $p=2$, $q=3$, $r=5$; write, as directed in the Rule,

1, $b+c$, $bb+bc+cc$, $bbcc+bc^3+c^4$, bbc^3+bc^4, bbc^4.

Multiply each term by a^{p-1}, which in our Example is a^{2-1}, or a; prefix to each Product the number of its *Permutations*, dividing at the ſame time the ſimilar Sums by $ſ^{p-1}$, $ſ^{p}$, $ſ^{p+1}$, &c. that is by $ſ$, $ſ^2$, $ſ^3$, &c; And the whole multiplied into $\frac{a}{ſ}$ will give the Expectation of $A=\frac{a}{ſ}$ into $\frac{a}{ſ}+\frac{2ab+2ac}{ſ^2}+\frac{3abb+6abc+3acc}{ſ^3}+\frac{12abbc+12abcc+4ac^3}{ſ^4}+\frac{20abbcc+20abc^3+5ac^4}{ſ^5}+\frac{60abbc^3+30abc^4}{ſ^6}+\frac{105abbc^4}{ſ^7}$.

If we now ſubſtitute for *a*, *b*, *c*, any numbers at pleaſure, we ſhall have the anſwer that belongs to thoſe ſuppoſed degrees of Skill. As if we make $a=1$, $b=1$, $c=1$; the Expectation of *A* will be, $\frac{1}{3}\times\frac{1}{3}+\frac{4}{9}+\frac{12}{27}+\frac{28}{81}+\frac{55}{243}+\frac{90}{729}+\frac{105}{2187}=\frac{1423}{2187}$. And, by like Operations, thoſe of *B* and *C* will be $\frac{635}{2187}$ and $\frac{119}{2187}$ reſpectively.

PROBLEM VII.

Two Gameſters A *and* B, *each having* 12 *Counters, play with three Dice, on condition that if* 11 *Points come up,* B *ſhall give one Counter to* A; *if* 14 *Points come up,* A *ſhall give one Counter to* B; *and that he ſhall be the winner who ſhall ſooneſt get all the Counters of his Adverſary: what is the Probability that each of them has of winning?*

SOLUTION.

Let the number of Counters which each of them has be $= p$; and let a and b be the number of Chances they have reſpectively for getting a Counter, each caſt of the Dice: which being ſuppoſed, I ſay that the Probabilities of winning are reſpectively as a^p to b^p; now becauſe in this caſe $p = 12$, and that, by the preceding Lemma, $a = 27$, and $b = 15$, it follows that the Probabilities of winning are reſpectively as 27^{12} to 15^{12}, or as 9^{12} to 5^{12}, or as 282429536481 to 244140625: which is the proportion aſſigned by *Huygens* in this particular caſe, but without any Demonſtration.

Or more generally:

Let p be the number of the Counters of A, and q the number of the Counters of B; and let the proportion of the Chances be as a to b. I ſay that the Probabilities of winning will be reſpectively as $a^q \times \overline{a^p - b^p}$ to $b^p \times \overline{a^q - b^q}$; and conſequently the Probabilities themſelves will be $\frac{a^q \times \overline{a^p - b^q}}{a^{p+q} - b^{p+q}} = R$, and $\frac{b^p \times \overline{a^q - b^q}}{a^{p+q} - b^{p+q}}. = S$.

DEMONSTRATION.

Let it be ſuppoſed that A has the Counters E, F, G, H, &c. whoſe number is p, and that B has the Counters I, K, L, &c. whoſe number is q: moreover, let it be ſuppoſed that the Counters are the thing played for, and that the value of each Counter is to the value of the following as a to b, in ſuch manner as that E, F, G, H, I, K, L be in geometric Progreſſion; this being ſuppoſed, A and B in every circumſtance of their Play may lay down two ſuch Counters as may be proportional to the number of Chances each has to get a ſingle Counter; for in the beginning of the Play, A may lay down the Counter H, which is the loweſt of his Counters, and B the Counter I, which is his higheſt; but H, I : : a, b, therefore A and B play upon equal terms. If A beats B, then A may lay down the Counter I which he has juſt got of his adverſary, and B the Counter K; but I, K : : a, b, therefore A and B ſtill play upon equal terms. But if A loſe the firſt time, then A may lay down the Counter G, and B the Counter H, which he juſt now got of his adverſary; but G, H : : a, b, and therefore they ſtill play upon equal terms as before: So that, as long as they play together, they play without advantage or diſadvantage. Now the Value of the Expectation which A has of getting all the Counters of B is the product of the Sum he expects to win, and of the probability of obtaining it, and the ſame holds alſo in reſpect to *B*: but the Expectations of *A* and *B* are ſuppoſed equal, and therefore the Probabilities which they have reſpectively of winning, are reciprocally proportional to the Sums they expect to win, that is, are directly proportional to the Sums they are poſſeſſed of. Whence the Probability which *A* has of winning all the Counters of *B*, is to the Probability which *B* has of winning all the Counters of *A*, as the Sum of the terms, E, F, G, H, whoſe number is p, to the Sum of the terms I, K, L, whoſe number is q, that is as $a^q \times \overline{a^p - b^p}$ to $b^p \times \overline{a^q - b^q}$; as will eaſily appear if thoſe terms, which are in geometric Progreſſion, are actually ſummed up by the known Methods. Now the Probabilities of winning are not influenced by the Suppoſition here made of each Counter being to the following in the proportion of a to b; and therefore when thoſe Counters are ſuppoſed of equal value, or rather of no value, but ſerving only to mark the number of Stakes won or loſt on either ſide, the Probabilities of winning will be the ſame as we have aſſigned.

COROLLARY 1.

If we ſuppoſe both a and b in a ratio of equality, the expreſſions whereby the Probabilities of winning are determined will be reduced to the proportion of p to q: which will eaſily appear if thoſe expreſſions be both divided by $a - b$.

COROLLARY 2.

If *A* and *B* play together for a Guinea a Game, and *A* has but one ſingle Guinea to loſe, but *B* any number, let it be never ſo large; if *A* in each Game has the Chance of 2 to 1, he is more likely to win all the Stock of *B* than to loſe his ſingle Guinea; and juſt as likely, if the Stock of *B* were infinite

REMARK.

If p and q, or either of them be large numbers, it will be convenient to work by Logarithms.

Thus, if *A* and *B* play a Guinea a Stake, and the number of Chances which *A* has to win each ſingle Stake be 43, but the number of Chances, which *B* has to win it, be 40, and they oblige themſelves to play till ſuch time as 100 Stakes are won or loſt; (the number p being $= q = 100$, and therefore the Ratio ſought being $\overline{\frac{43}{40}}\Big|^{100}$.)

From the logarithm of 43 = 1.6334685
Subtract the logarithm of 40 = 1.6020600
Difference = 0.0314085

Multiply this Difference by the number of Stakes to be played off, *viz.* 100, the product will be 3.1408500, to which anſwers in the Table of Logarithms 1383; therefore the Odds that *A* beats *B* are 1383 to 1.

Now in all circumſtances wherein *A* and *B* venture an equal Sum, the Sum of the numbers expreſſing the Odds, is to their difference, as the Money played for, is to the Gain of the one, and the Loſs of the other.

Wherefore, multiplying 1382 difference of the numbers expreſſing the Odds by 100, which is the Sum ventured by each Man, and dividing the product by 1384, Sum of the Numbers expreſſing the Odds, the Quotient will be, within a trifle, 99 Guineas, and 2 Shillings, ſuppoſing Guineas at 21ſh.

If inſtead of ſuppoſing the proportion of the Chances whereby *A* and *B* may reſpectively win a Stake to be as 43 to 40, we ſuppoſe them as 44 to 40, or as 11 to 10, the Expectation of *A* will be worth above 99 Guineas, 20 Shillings and 1 Penny.

PROBLEM VIII.

Two Gameſters A *and* B *lay by* 24 *Counters, and play with three Dice, on this condition; that if* 11 *Points come up,* A *ſhall take one Counter out of the heap; if* 14, B *ſhall take out one; and he ſhall be reputed the winner who ſhall ſooneſt get* 12 *Counters.*

This Problem differs from the preceding in this, that the Play will be at an end in 23 Caſts of the Die at moſt; (that is, of thoſe Caſts which are favourable either to *A* or *B*) whereas in the preceding caſe the Counters paſſing continually from one hand to the other, it will often happen that *A* and *B* will be in ſome of the ſame circumſtances they were in before, which will make the length of the Play unlimited.

SOLUTION.

Taking a and b in the proportion of the Chances which there are to throw 11, and 14, let $a + b$ be raiſed to the 23^{d} Power, that is, to ſuch Power as is denoted by the number of all the Counters wanting one: then ſhall the 12 firſt terms of that Power be to the 12 laſt in the ſame proportion as are the Probabilities of winning.

PROBLEM IX.

Suppoſing A *and* B, *whoſe proportion of ſkill is as* a *to* b, *to play together, till* A *either wins the number* q *of Stakes, or loſes the number* p *of them; and that* B *ſets at every Game the Sum* G *to the Sum* L; *it is required to find the Advantage or Diſadvantage of* A.

SOLUTION.

Firſt, Let the number of Stakes to be won or loſt on either ſide be equal, and let that number be p; let there be alſo an equality of ſkill between the Gameſters: then I ſay that the Gain of A will be $pp \times \frac{G-L}{2}$, that is the ſquare of the number of Stakes which either Gameſter is to win or loſe, multiplied by one half of the difference of the value of the Stakes. Thus if A and B play till ſuch time as ten Stakes are won or loſt, and B ſets one and twenty Shillings to 20; then the Gain of A will be 100 times the half difference between 21 and 20 Shillings, *viz.* 50 ſh.

Secondly, Let the number of Stakes be unequal, ſo that A be obliged either to win the number q of Stakes, or to loſe the number p; let there be alſo an equality of Chance between A and B: then I ſay that the Gain of A will be $pq \times \frac{G-L}{2}$; that is the Product of the two numbers of Stakes, and one half the difference of the value of the Stakes multiplied together. Thus if A and B play together till ſuch time as either A wins eight Stakes or loſes twelve, then the Gain of A will be the product of the two numbers 8 and 12, and of $6^{d.}$ half the difference of the Stakes, which product makes $2^{L.}$ 8 ſh.

Thirdly, Let the number of Stakes be equal, but let the number of Chances to win a Game, or the Skill of the Gameſters be unequal, in the proportion of a to b; then I ſay that the Gain of A will be $\frac{pa^p - pb^p}{a^p + b^p} \times \frac{aG - bL}{a - b}$.

Fourthly, Let the number of Stakes be unequal, and let alſo the number of Chances be unequal: then I ſay that the Gain of A will be $\frac{qa^q \times a^p - b^p - pb^p \times a^q - b^q}{a^{p+q} - b^{p+q}}$ multiplied by $\frac{aG-bL}{a-b}$.

DE-

DEMONSTRATION.

Let R and S reſpectively repreſent the Probabilities which *A* and *B* have of winning all the Stakes of their Adverſary; which Probabilities have been determined in the vii^th^ *Problem.* Let us firſt ſuppoſe that the Sums depoſited by *A* and *B* are equal, *viz.* G, and G: now ſince *A* is either to win the Sum qG, or loſe the Sum pG, it is plain that the Gain of *A* ought to be eſtimated by RqG — SpG; moreover ſince the Sums depoſited are G and G, and that the proportion of the Chances to win one Game is as a to b, it follows that the Gain of *A* for each individual Game is $\frac{aG-bG}{a+b}$; and for the ſame reaſon the Gain of each individual Game would be $\frac{aG-bL}{a+b}$, if the Sums depoſited by *A* and *B* were reſpectively L and G. Let us therefore now ſuppoſe that they are L and G; then in order to find the whole Gain of *A* in this ſecond circumſtance, we may conſider that whether *A* and *B* lay down equal Stakes or unequal Stakes, the Probabilities which either of them has of winning all the Stakes of the other, ſuffer not thereby any alteration, and that the Play will continue of the ſame length in both circumſtances before it is determined in favour of either; wherefore the Gain of each individual Game in the firſt caſe, is to the Gain of each individual Game in the ſecond, as the whole Gain of the firſt caſe, to the whole Gain of the ſecond; and conſequently the whole Gain of the ſecond caſe will be $\overline{Rq-Sp} \times \frac{aG-bL}{a-b}$, or reſtoring the values of R and S, $\frac{qa^q \times \overline{a^p-b^p} - pb^p \times \overline{a^q-b^q}}{a^{p+q}-b^{p+q}}$ multiplied by $\frac{aG-bL}{a-b}$.

PROBLEM X.

Three Perſons A, B, C, *out of a heap of* 12 *Counters, whereof* 4 *are white, and* 8 *black, draw blindfold one Counter at a time, in this manner*; A *begins to draw*; B *follows* A; C *follows* B; *then* A *begins again*; *and they continue to draw in the ſame order, till one of them who is to be reputed the winner, draws the firſt white. What are the reſpective Probabilities of their winning?*

SOLUTION.

Let n be the number of all the Counters, a the number of white, b the number of black, and 1 the whole Stake or the Sum played for.

1°. Since A has a Chances for a white Counter, and b Chances for a black, it follows that the Probability of his winning is $\frac{a}{a+b} = \frac{a}{n}$; therefore the Expectation he has upon the Stake 1, ariſing from the circumſtance he is in, when he begins to draw, is $\frac{a}{n} \times 1 = \frac{a}{n}$: let it therefore be agreed among the Adventurers, that A ſhall have no Chance for a white Counter, but that he ſhall be reputed to have had a black one, which ſhall actually be taken out of the heap, and that he ſhall have the Sum $\frac{a}{n}$ paid him out of the Stake, for an Equivalent. Now $\frac{a}{n}$ being taken out of the Stake there will remain $1 - \frac{a}{n} = \frac{n-a}{n} = \frac{b}{n}$.

2°. Since B has a Chances for a white Counter, and that the number of remaining Counters is $n - 1$, his Probability of winning will be $\frac{a}{n-1}$; whence his Expectation upon the remaining Stake $\frac{b}{n}$, ariſing from the circumſtance he is now in, will be $\frac{ab}{n \cdot n-1}$. Suppoſe it therefore agreed that B ſhall have the Sum $\frac{ab}{n \cdot n-1}$ paid him out of the Stake, and that a black Counter ſhall alſo be taken out of the heap. This being done, the remaining Stake will be $\frac{b}{n} - \frac{ab}{n \cdot n-1}$ or $\frac{nb-b-ab}{n \cdot n-1}$, but $nb - ab = bb$; wherefore the remaining Stake is $\frac{b \cdot b-1}{n \cdot n-1}$.

3°. Since C has a Chances for a white Counter, and that the number of remaining Counters is $n-2$, his Probability of winning will be $\frac{a}{n-2}$, and therefore his Expectation upon the remaining Stake ariſing from the circumſtance he is now in, will be $\frac{b \cdot b-1 \cdot a}{n \cdot n-1 \cdot n-2}$, which we will likewiſe ſuppoſe to be paid him out of the Stake, ſtill ſuppoſing a black Counter taken out of the heap.

4°. A may have out of the remainder the Sum $\frac{b \cdot b-1 \cdot b-2 \cdot a}{n \cdot n-1 \cdot n-2 \cdot n-3}$; and ſo of the reſt till the whole Stake be exhauſted.

Wherefore having written the following general Series; *viz.* $\frac{a}{n} + \frac{b}{n-1}P + \frac{b-1}{n-2}Q + \frac{b-2}{n-3}R + \frac{b-3}{n-4}S$, &c. wherein P, Q, R, S, &c. denote the preceding Terms, take as many Terms of this Series as there are Units in $b+1$, (for ſince b repreſents the number of black Counters, the number of drawings cannot exceed $b+1$,) then take for *A* the firſt, fourth, ſeventh, &c. Terms; take for *B* the ſecond, fifth, eighth, &c. for *C* the third, ſixth, &c. and the Sums of thoſe Terms will be the reſpective Expectations of *A*, *B*, *C*; or becauſe the Stake is fixed, theſe Sums will be proportional to the reſpective Probabilities of winning.

Now to apply this to the preſent caſe, make $n = 12$, $a = 4$, $b = 8$, and the general Series will become $\frac{4}{12} + \frac{8}{11}P + \frac{7}{10}Q + \frac{6}{9}R + \frac{5}{8}S + \frac{4}{7}T + \frac{3}{6}U + \frac{2}{5}X + \frac{1}{4}Y$: or multiplying the whole by 495 to take away the fractions, the Series will be $165 + 120 + 84 + 56 + 35 + 20 + 10 + 4 + 1$.

Therefore aſſigning to *A* $165 + 56 + 10 = 231$, to *B* $120 + 35 + 4 = 159$, to *C* $84 + 20 + 1 = 105$, the Probabilities of winning will be proportional to the numbers 231, 159, 105, or 77, 53, 35.

If there be never ſo many Gameſters *A*, *B*, *C*, *D*, &c. whether they take every one of them one Counter or more; or whether the ſame or a different number of Counters; the Probabilities of winning will be determined by the ſame general Series.

REMARK I.

The preceding Series may in any particular caſe be ſhortened; for if $a = 1$, then the Series will be $\frac{1}{n} \times \overline{1+1+1+1+1+1+1}$, &c.

Hence it may be obſerved, that if the whole number of Counters be exactly diviſible by the number of Perſons concerned in the Play, and that there be but one ſingle white Counter in the whole, there will be no advantage or diſadvantage to any one of them from the ſituation he is in, in reſpect to the order of drawing.

If $a = 2$, then the Series will be $\frac{2}{n \cdot \overline{n-1}} \times \overline{\overline{n-1} + \overline{n-2} + \overline{n-3} + \overline{n-4} + \overline{n-5}}$, &c.

If $a = 3$, then the Series will be $\frac{3}{n \cdot \overline{n-1} \cdot \overline{n-2}} \times \overline{\overline{n-1} \cdot \overline{n-2} + \overline{n-2} \cdot \overline{n-3} + \overline{n-3} \cdot \overline{n-4}}$, &c.

If $a = 4$, then the Series will be $\frac{4}{n \cdot \overline{n-1} \cdot \overline{n-2} \cdot \overline{n-3}}$ $\times\, \overline{n-1} \cdot \overline{n-2} \cdot \overline{n-3} + \overline{n-2} \cdot \overline{n-3} \cdot \overline{n-4}$, &c.

Wherefore rejecting the common Multiplicators; the ſeveral Terms of theſe Series taken in due order, will be proportional to the ſeveral Expectations of any number of Gameſters: thus in the caſe of this Problem where $n = 12$, and $a = 4$, the Terms of the Series will be,

For *A*	For *B*	For *C*
$11 \times 10 \times 9 = 990$	$10 \times 9 \times 8 = 720$	$9 \times 8 \times 7 = 504$
$8 \times 7 \times 6 = 336$	$7 \times 6 \times 5 = 210$	$6 \times 5 \times 4 = 120$
$5 \times 4 \times 3 = 60$	$4 \times 3 \times 2 = 24$	$3 \times 2 \times 1 = 6$
1386	954	630

Hence it follows that the Probabilities of winning will be reſpectively as 1386, 954, 630, or dividing all by 18, as 77, 53, 35, as had been before determined.

REMARK 2.

But if the Terms of the Series are many, it will be convenient to ſum them up by means of the following Method, which is an immediate conſequence of the fifth Lemma of Sir *Iſaac Newton's Principia*, Book III; and of which the Demonſtration may be deduced from his *Analyſis*.

If there be a Series of Terms, A, B, C, D, E, &c. let each Term be ſubtracted from that which immediately follows it, and let the Remainders be called firſt Differences, then ſubtract each difference from that which immediately follows it, and let the remainders be called ſecond differences; again, let each ſecond difference be ſubtracted from that which immediately follows it, and let the remainders be called third differences, and ſo on. Let the firſt of the firſt Differences be called $\overset{\prime}{d}$, the firſt of the ſecond $\overset{\prime\prime}{d}$, the firſt of the third $\overset{\prime\prime\prime}{d}$, &c. and let x be the interval between the firſt Term A, and any other Term, ſuch as E, that is, let the number of Terms from A to E, both incluſive, be $x + 1$, then the Term $E = A + x\overset{\prime}{d} + \frac{x}{1} \times \frac{x-1}{2}\overset{\prime\prime}{d} + \frac{x \times \overline{x-1} \times \overline{x-2}}{1 \cdot 2 \cdot 3}\overset{\prime\prime\prime}{d}$, &c. From hence it manifeſtly follows, that let the number of Terms between A and E be never ſo great, if it ſo happen that all the differences of one of the orders are equal to one another, the following differences of all the other orders will all be $= 0$; and that therefore the laſt Term will be aſſignable by ſo many Terms only of the Series above-written, as are denoted by the firſt Difference that happens to be $= 0$.

This being premiſed, it will be eaſy to ſhew, how the Sums of thoſe Terms may be taken; for if we imagine a new Series whereof the firſt Term ſhall be $= 0$; the ſecond $= A$; the third $= A + B$; the fourth $= A + B + C$; the fifth $= A + B + C + D$, and ſo on; it is plain that the aſſigning one Term of the new Series is finding the Sum of all the Terms A, B, C, D, &c. Now ſince thoſe Terms are the differences of the Sums 0, A, A + B, A + B + C, A + B + C + D, &c. and that by Hypotheſis ſome of the differences of A, B, C, D, are $= 0$, it follows that ſome of the differences of the Sums will alſo be $= 0$; and that whereas in the Series $A + x\overset{\prime}{d} + \frac{x}{1} \times \frac{x-1}{2}\overset{\prime\prime}{d}$, &c. whereby a Term was aſſigned, A repreſented the firſt Term, $\overset{\prime}{d}$ the firſt of the firſt differences, $\overset{\prime\prime}{d}$ the firſt of the ſecond differences, and that x repreſented the Interval between the firſt Term and the laſt, we are now to write 0 inſtead of A; A inſtead of $\overset{\prime}{d}$; $\overset{\prime}{d}$ inſtead of $\overset{\prime\prime}{d}$; $\overset{\prime\prime}{d}$ inſtead of $\overset{\prime\prime\prime}{d}$, &c. and $x + 1$ inſtead of x; which being done the Series expreſſing the Sums will be $\underline{0 + \overline{x+1} \times A + \frac{x+1 \,.\, x}{1 \,.\, 2}\overset{\prime}{d} + \frac{x+1 \,.\, x \,.\, x-1}{1 \,.\, 2 \,.\, 3}\overset{\prime\prime}{d}}$, &c. or $\overline{x+1} \times A + \frac{x}{2}\overset{\prime}{d} + \frac{x \,.\, x-1}{2 \,.\, 3}\overset{\prime\prime}{d} + \frac{x \,.\, x-1 \,.\, x-2}{2 \,.\, 3 \,.\, 4}\overset{\prime\prime\prime}{d}$, &c. where it will not perhaps be improper to take notice, that the Series by me exhibited in my firſt Edition, though ſeemingly differing from this, is the ſame at bottom.

But to apply this to a particular caſe, let us ſuppoſe that three Perſons *A*, *B*, *C* playing on the ſame conditions as are expreſſed in this xth *Problem*, the whole number of Counters were 100, inſtead of 12, ſtill preſerving the ſame number 4 of white Counters, and that it were required to determine the Expectations of *A*, *B*, *C*.

It is plain from what has been ſaid in the firſt Remark, that the Expectation of *A* will be proportional to the ſum of the numbers

$99 \times 98 \times 97 + 96 \times 95 \times 94 + 93 \times 92 \times 91 + 90 \times 89 \times 88$, &c.

that the Expectation of B will be proportional to the Sum of the numbers

$$98\times97\times96+95\times94\times93+92\times91\times90+89\times88\times87,\ \&c.$$

and laſtly, that the Expectation of C will be proportional to the Sum of the numbers

$$97\times96\times95+94\times93\times92+91\times90\times89+88\times87\times86,\ \&c.$$

But as the number of Terms which conſtitute thoſe three Series is equal to the number of black Counters increaſed by 1, as it has been obſerved before, it follows that the number of all the Terms diſtributed among A, B, C, muſt be 97; now dividing 97 by the number of Gameſters which in this caſe is 3, the quotient will be 32; and there remaining 1 after the diviſion, it is an indication that 33 Terms enter the Expectation of A, that 32 Terms enter the Expectation of B, and 32 likewiſe the Expectation of C; from whence it follows that the laſt Term of thoſe which belong to A will be $3\times2\times1$, the laſt of thoſe which belong to B will be $5\times4\times3$, and the laſt of thoſe which belong to C will be $4\times3\times2$.

And therefore if we invert the Terms, making that the firſt which was the laſt, and take the differences, according to what has been preſcribed, as follows;

$$\begin{array}{lllr|r|r|r}
 & & & A & & & \\
 & & & & \acute{d} & & \\
3\times & 2\times & 1= & 6 & & \ddot{d} & \\
 & & & & 114 & & \overset{\prime\prime\prime}{d} \\
6\times & 5\times & 4= & 120 & & 270 & \\
 & & & & 384 & & 162 \\
9\times & 8\times & 7= & 504 & & 432 & \\
 & & & & 816 & & 162. \\
12\times & 11\times & 10= & 1320 & & 594 & \\
 & & & & 1410 & & \\
15\times & 14\times & 13= & 2730 & & & \\
\&c. & & & & & &
\end{array}$$

then the Expectation of A, as deduced from the general Theorem, will be expreſſed by

$$\overline{x+1}\times\overline{6+\frac{114x}{2}+\frac{x.x-1}{2.3}\times270+\frac{x.x-1.x-2}{2.3.4}\times162}:$$

which being contracted, then reduced into its factors, will be equivalent to

$$\frac{3}{4}\times\overline{x+1}\times\overline{x+2}\times\overline{3x+1}\times\overline{3x+4}.$$

In like manner, it will be found that the Expectation of B is equivalent to

$$\frac{3}{4}\times\overline{x+1}\times\overline{x+2}\times\overline{3x+5}\times\overline{3x+8}.$$

And that the Expectation of *C* is equivalent to

$$\frac{3}{4} \times \overline{x+1} \times \overline{x+2} \times \overline{9xx+27x+16}.$$

Now x in each caſe repreſents the number of Terms wanting one, which belong ſeverally to *A*, *B*, *C*; wherefore making $x+1=p$, the ſeveral Expectations will now be expreſſed by the number of Terms which were originally to be ſummed up, and which will be as folllows.

For *A*, $p \times \overline{p+1} \times \overline{3p-2} \times \overline{3p+1}$
For *B*, $p \times \overline{p+1} \times \overline{3p+2} \times \overline{3p+5}$
For *C*, $p \times \overline{p+1} \times \overline{9pp+9p-2}$

But ſtill it is to be conſidered, that p in the firſt caſe anſwers to the number 33, and in the other two caſes to 32; and therefore p being interpreted for the ſeveral caſes as it ought to be, the ſeveral Expectations will be found proportional to the numbers 41225, 39592, 38008.

If the number of all the Counters were 500, and the number of the white ſtill 4, then the number of all the Terms repreſenting the Expectations of *A*, *B*, *C* would be 497: now this number being divided by 3, the Quotient is 165, and the remainder 2. From whence it follows that the Expectations of *A* and *B* conſiſt of 166 terms each, and the Expectation of *C* only of 165, and therefore the loweſt Term of all, *viz.* 3 × 2 × 1 will belong to *B*, the laſt but one 4 × 3 × 2 will belong to *A*, and the laſt but two will belong to *C*.

PROBLEM XI.

If A, B, C *throw in their turns a regular Ball having* 4 *white faces and eight black ones; and he be to be reputed the winner who ſhall firſt bring up one of the white faces; it is demanded, what the proportion is of their reſpective Probabilities of winning?*

SOLUTION.

The Method of reaſoning in this Problem is exactly the ſame as that which we have made uſe of in the Solution of the preceding: but whereas the different throws of the Ball do not diminiſh the number of its Faces; in the room of the quantities $b-1$, $b-2$,

$b-3$, &c. $n-1$, $n-2$, $n-3$, &c. employed in the Solution of the aforeſaid Problem, we muſt ſubſtitute b and n reſpectively, and the Series belonging to that Problem will be changed into the following, which we ought to conceive continued infinitely.

$$\frac{a}{n}+\frac{ab}{nn}+\frac{abb}{n^3}+\frac{ab^3}{n^4}+\frac{ab^4}{n^5}+\frac{ab^5}{n^6}, \text{ \&c.}$$

then taking every third Term thereof, the reſpective Expectations of *A, B, C* will be expreſſed by the following Series,

$$\frac{a}{n}+\frac{ab^3}{n^4}+\frac{ab^6}{n^7}+\frac{ab^9}{n^{10}}+\frac{ab^{12}}{n^{13}}, \text{ \&c.}$$
$$\frac{ab}{nn}+\frac{ab^4}{n^5}+\frac{ab^7}{n^8}+\frac{ab^{10}}{n^{11}}+\frac{ab^{13}}{n^{14}}, \text{ \&c.}$$
$$\frac{abb}{n^3}+\frac{ab^5}{n^6}+\frac{ab^8}{n^9}+\frac{ab^{11}}{n^{12}}+\frac{ab^{14}}{n^{15}}, \text{ \&c.}$$

But the Terms, whereof each Series is compoſed, are in geometric Progreſſion, and the ratio of each Term in each Series to the following is the ſame; wherefore the Sums of theſe Series are in the ſame proportion as their firſt Terms, *viz.* as $\frac{a}{n}$, $\frac{ab}{nn}$, $\frac{abb}{n^3}$, or as nn, bn, bb; that is, in the preſent caſe, as 144, 96, 64, or 9, 6, 4. Hence the reſpective Probabilities of winning will likewiſe be as the numbers 9, 6, 4.

COROLLARY 1.

If there be any other number of Gameſters *A, B, C, D,* &c. playing on the ſame conditions as above, take as many Terms in the proportion of n to b, as there are Gameſters, and thoſe Terms will reſpectively denote the ſeveral Expectations of the Gameſters.

COROLLARY 2.

If there be any number of Gameſters *A, B, C, D,* &c. playing on the ſame conditions as above, with this difference only, that all the Faces of the Ball ſhall be marked with particular figures 1, 2, 3, 4, &c. and that a certain number p of thoſe Faces ſhall intitle *A* to be the winner; and that likewiſe a certain number of them, as q, r, s, t, &c. ſhall reſpectively intitle *B, C, D, E,* &c. to be winners: make $n-p=a$, $n-q=b$, $n-r=c$, $n-s=d$, $n-t=e$, &c. then in the following Series;

$$\frac{p}{n}+\frac{qa}{nn}+\frac{rab}{n^3}+\frac{sabc}{n^4}\quad\frac{tabcd}{n^5}, \text{ \&c.}$$

the Terms taken in due order will reſpectively repreſent the ſeveral Probabilities of winning.

For if the law of the Play be ſuch, that every Man having once played in his turn, ſhall begin regularly again in the ſame manner, and that continually, till ſuch time as one of them wins; we are to take as many Terms of the Series as there are Gameſters, and thoſe Terms will repreſent the reſpective Probabilities of winning.

But the Reaſon of this Rule will beſt appear if we apply it to ſome eaſy Example.

Let therefore the three Gameſters *A*, *B*, *C* throw a Die of 12 faces in their Turns; of which 5 faces are favourable to *A*, 4 faces are favourable to *B*, and the remaining 3 give the Stake to *C*. Then $p = 5$, $q = 4$, $r = 3$: and there being but 3 Gameſters, the ſame Chances, and in the ſame Order *A*, *B*, *C*, will recur perpetually after a Round of three throws, till the Stake is won; or rather, as we ſuppoſe in the demonſtration, till the Stake is totally exhauſted, by each Gameſter, inſtead of his throw, taking out of it the part to which the chance of that throw entitles him.

Now *A* having p Chances out of n, or 5 out of 12, to get the whole Stake at the firſt Throw, let him take out of it the Value of this Chance $\frac{p}{n}$; and there will remain $1 - \frac{p}{n} = \frac{n-p}{n} = \frac{a}{n}$ to be thrown for by *B*.

And *B*'s Chances for winning in his Throw being q out of n, or 4 out of 12, the Value of his preſent Expectation is $\frac{q}{n} \times \frac{a}{n} = \frac{qa}{n^2}$; which if he takes out of the Stake $\frac{a}{n}$ there will remain $\frac{a}{n} - \frac{qa}{n^2} = \frac{a}{n} \times \overline{1 - \frac{q}{n}} = \frac{a}{n} \times \frac{b}{n}$, to be thrown for by *C*.

His Chances for getting this Stake being r out of n, or 3 out of 12, the Value of his Expectation is $\frac{rab}{n^3}$; which he may take out of the Stake $\frac{ab}{n^2}$: and reſign the Die to *A*, who begins the ſecond Round.

But if, for the Stakes that remain after the firſt, ſecond, third, &c. Rounds, we write R', R'', R''', &c. reſpectively, it is manifeſt that the Value of a Gameſter's Chance in each Round is proportional to the Stake R', R'', R'', &c. which remained at the beginning of that Round. Thus the Value of *A*'s firſt Throw having been $\frac{p}{n} \times 1$, the Value of his ſecond will be $\frac{p}{n} \times R'$, of his third, $\frac{p}{n} \times R''$, &c. And the Value of *B*'s firſt Throw having been $\frac{qa}{n^2} \times 1$, that of his ſecond will $\frac{qa}{n^2} \times R'$, of his third, $\frac{qa}{n^2} \times R''$, &c. and the like for the ſeveral Expectations of *C*.

Put $S = 1 + R' + R'' + R'''$, &c. and the Total of *A*'s Expectations will be $\frac{p}{n} \times S$; of *B*, $\frac{qa}{n^2} \times S$; of *C*, $\frac{rab}{n^3} \times S$: or rejecting the common Factor *S*, the Expectations of *A*, *B*, *C*, at the beginning of the Play will be as $\frac{p}{n}$, $\frac{qa}{n^2}$, $\frac{rab}{n^3}$, reſpectively: that is as the 3 firſt Terms of the Series. And the like reaſoning will hold, be the Number of Gameſters, their favourable Chances, or order of Throwing, what you will.

In the preſent Example, $\frac{p}{n} = \frac{5}{12} = \frac{720}{1728}$; $\frac{qa}{n^2} = \frac{28}{144} = \frac{336}{1728}$; $\frac{rab}{n^3} = \frac{168}{1728}$: and the Chances of *A*, *B*, *C*, reſpectively, are as the Numerators 720, 336, 168; that is, as 30, 14, 7. or the whole Stake being 51 pieces, *A* can claim 30 of them, *B* 14, and *C* the remaining 7.

In making up this Stake, the Gameſters *A*, *B*, *C*, were, at equal play, to contribute only in proportion to their Chances of winning; that is in the proportion of p, q, r, or 5, 4, 3, reſpectively: and, before the Order of throwing was fixt, their Chances muſt have been exactly worth what they paid in to the Stake. What gives *A* the great advantage now is, an antecedent good luck of being the firſt to throw. If *B* had been the firſt; or if *A*, taking his firſt Throw, had miſt of a p face, then *B*'s Chance had been the better of the two.

And if it were the Law of Play that every Man ſhould play ſeveral times together, for inſtance twice: then taking for *A* the two firſt Terms, for *B* the two following, and ſo on; each couple of Terms will repreſent the reſpective Probabilities of winning, obſerving now that p and q are equal, as alſo r and s.

But if the Law of Play ſhould be irregular, then you are to take for each Man as many Terms of the Series as will anſwer that irregularity, and continue the Series till ſuch time as it gives a ſufficient Approximation.

Yet if, at any time, the Law of the Play having been irregular, ſhould afterwards recover its regularity, the Probabilities of winning, will (with the help of this Series) be determined by finite expreſſions.

Thus if it ſhould be the Law of the Play, that two Men *A* and *B* having played irregularly for ten times together, tho' in a manner agreed on between them, they ſhould alſo agree that after ten throws, they ſhould play alternately each in his turn: diſtribute the ten firſt Terms of the Series between them, according to the order fixed upon by their convention, and having ſubtracted the Sum of thoſe Terms from Unity, divide the remainder of it between them in the proportion of the two following Terms, which add reſpectively to the Shares they had before; then the two parts of Unity which *A* and *B* have thus obtained, will be proportional to their reſpective Probabilities of winning.

PROBLEM XII.

There are any number of Gameſters, who in their Turns, which are decided by Lots, turn a Cube, having 4 *of its Faces marked* T, P, D, A, *the other two Faces which are oppoſite have each a little Knob or Pivet, about which the Cube is made to turn; the Gameſters each lay down a Sum agreed upon, the firſt begins to turn the Cube; now if the Face* T *be brought up, he ſweeps all the Money upon the Board, and then the Play begins anew; if any other Face is brought up, he yields his place to the next Man, but with this difference, that if the Face* P *comes up, he, the firſt Man, puts down as much Money as there was upon the Board; if the Face* D *comes up, he neither takes up any Money nor lays down any; if the Face* A *comes up, he takes up half of the Money upon the Board; when every Man has played in his Turn upon the ſame conditions as above, there is a recurrency of Order, whereby the Board may be very much enlarged,* viz. *if it ſo happen that the Face* T *is intermitted during many Trials: now the Queſtion is this; when a Gameſter comes to his Turn, ſuppoſing him afraid of laying down as much Money as there is already, which may be conſiderable, how muſt he compound for his Expectation with a Spectator willing to take his place.*

SOLUTION.

Let us ſuppoſe for a little while that the number of Gameſters is infinite, and that what is upon the Board is the Sum s; then, there being 1 Chance in 4 for the Face T to come up, it follows that the Expectation of the firſt Man, upon that ſcore, is $\frac{1}{4}$ 2°. There being 1 Chance in 4 for the Face P to come up, whereby he would neceſſarily loſe $\int$, (by reaſon that the number of Gameſters having been ſuppoſed infinite, his Chance of playing would never return again) it follows that his Loſs upon that account ought to be eſtimated by $\frac{1}{4}\int$. 3°. There being 1 Chance in 4 for the Face D to come up, whereby he would neither win or loſe any thing, we may proceed to the next Chance. 4°. There being 1 Chance in 4 for the Face A to come up, which intitles him to take up $\frac{1}{2}\int$, his Expectation, upon that account, is $\frac{1}{8}\int$, or ſuppoſing $8 = n$, his Expectation is $\frac{1}{n}\int$; now out of the four caſes abovementioned the firſt and ſecond do deſtroy one another, the third neither contributes to Gain or Loſs, and therefore the clear Gain of the firſt Man is upon account of the fourth Caſe; let it therefore be agreed among the Adventurers, that the firſt Man ſhall not try his Chance, but that he ſhall take the Sum $\frac{1}{n}\int$ out of the common Stake $\int$, and that he ſhall yield his Turn to the next Man.

But before I proceed any farther, it is proper to prevent an Objection that may be made againſt what I have aſſerted above, *viz.* that the Face D happening to come up, the Adventurer in that caſe would loſe nothing, becauſe it might be ſaid that the number of Gameſters being infinite, he would neceſſarily loſe the Stake he has laid down at firſt; but the anſwer is eaſy, for ſince the number of particular Stakes is infinite, and that the Sum of all the Stakes is ſuppoſed only equal to $\int$, it follows that each particular Stake is nothing in compariſon to the common Stake $\int$, and therefore that common Stake may be looked upon as a preſent made to the Adventurers. Now to proceed; I ſay that the Sum $\frac{1}{n}\int$ having been taken out of the common Stake $\int$, the remaining Stake will be $\frac{n-1}{n}\int$ or $\frac{d}{\int}$, ſuppoſing $n - 1 = d$: but by reaſon that the firſt Man was allowed $\frac{1}{n}$ part of the common Stake, ſo ought the next Man to be allowed $\frac{1}{n}$ part of the preſent Stake $\frac{d}{n}\int$, which will make it that the Expectation of the ſecond Man will be $\frac{d}{nn}\int$; Again, the Expectation of the ſecond Man being to the Expectation of the firſt as $\frac{d}{n}$ to 1, the Expectation of the third muſt be to the Expectation of the ſecond alſo as $\frac{d}{n}$ to 1, from whence it follows that the Expectation of the third Man will be $\frac{dd}{n^3}ſ$, and the Expectation of the fourth $\frac{d^3}{n^4}ſ$, and ſo on; which may fitly be repreſented by the Series $ſ$ *into* $\overset{1}{\frac{1}{n}} + \overset{2}{\frac{d}{nn}} + \overset{3}{\frac{dd}{n^3}} + \overset{4}{\frac{d^3}{n^4}} + \overset{5}{\frac{d^5}{n^5}} + \overset{6}{\frac{d^6}{n^6}}$, &c. Now the Sum of that infinite Series, which is a Geometric Progreſſion, is $\frac{ſ}{n-d}$, but d having been ſuppoſed $=n-1$, then $n-d=1$, and therefore the Sum of all the Expectations is only $ſ$, as it ought to be.

Now let us ſuppoſe that inſtead of an infinite number of Gameſters, there are only two; then, in this caſe, we may imagine that the firſt Man has the *firſt*, *third*, *fifth*, *ſeventh* Terms of that Series, and all thoſe other Terms *in infinitum* which belong to the *odd* places, and that the ſecond Man has all the Terms which belong to the *even* places; wherefore the Expectation of the firſt Man is $\frac{ſ}{n}$ *into* $1 + \frac{dd}{nn} + \frac{d^4}{n^4} + \frac{d^6}{n^6} + \frac{d^3}{n^8}$, &c. and the Expectation of the ſecond is $\frac{dſ}{nn}$ *into* $1 + \frac{dd}{nn} + \frac{d^4}{n^4} + \frac{d^6}{n^6} + \frac{d^8}{n^8}$, &c. and therefore the Ratio of their Expectations is as $\frac{ſ}{n}$ to $\frac{dſ}{nn}$, or as 1 to $\frac{d}{n}$, that is as n to $n-1$, or as 8 to 7; and therefore the Expectation of the firſt Man is $\frac{8}{15}ſ$, and the Expectation of the ſecond Man is $\frac{7}{15}ſ$; and therefore if a Spectator has a mind to take the place of the firſt Man, he ought to give him $\frac{8}{15}ſ$.

But if the number of Gameſters be three, take a third proportional to n and d, which will be $\frac{dd}{n}$, and therefore the three Expectations will be reſpectively proportional to n, d, $\frac{dd}{n}$, or to nn, dn, dd, and therefore the Expectation of the firſt Man is $\frac{nn}{nn+dn+dd}ſ$ which in this caſe is $= \frac{64}{169}ſ$.

Univerſally, Let p be the number of Adventurers, then the Sum for which the Expectation of the firſt Man may be transferred to another is $\frac{n^{p-1}}{n^p - d^p}ſ$.

The Game of BASSETTE.

Rules of the Play.

The Dealer, otherwiſe called the *Banker*, holds a pack of 52 Cards, and having ſhuffled them, he turns the whole pack at once, ſo as to diſcover the laſt Card; after which he lays down by couples all the Cards.

The Setter, otherwiſe called the *Ponte*, has 13 Cards in his hand, one of every ſort, from the King to the Ace, which 13 Cards are called a *Book*; out of this Book he takes one Card or more at pleaſure, upon which he lays a Stake.

The Ponte may at his choice, either lay down his Stake before the pack is turned, or immediately after it is turned; or after any number of Couples are drawn.

The firſt caſe being particular, ſhall be calculated by itſelf; but the other two being comprehended under the ſame Rules, we ſhall begin with them.

Suppoſing the Ponte to lay down his Stake after the Pack is turned, I call 1, 2, 3, 4, 5, &c. the places of thoſe Cards which follow the Card in view, either immediately after the pack is turned, or after any number of couples are drawn.

If the Card upon which the Ponte has laid a Stake comes out in any odd place, except the firſt, he wins a Stake equal to his own.

If the Card upon which the Ponte has laid a Stake comes out in any even place, except the ſecond, he loſes his Stake.

If the Card of the Ponte comes out in the firſt place, he neither wins nor loſes, but takes his own Stake again.

If the Card of the Ponte comes out in the ſecond place, he does not loſe his whole Stake, but only a part of it, *viz.* one half, which to make the Calculation more general we ſhall call y. In this caſe the Ponte is ſaid to be *Faced.*

When the Ponte chuſes to come in after any number of Couples are down; if his Card happens to be but once in the Pack, and is the very laſt of all, there is an exception from the general Rule; for tho' it comes out in an odd place, which ſhould intitle him to win a Stake equal to his own, yet he neither wins nor loſes from that circumſtance, but takes back his own Stake.

PRO-

PROBLEM XIII.

To eſtimate at Baſſette *the Loſs of the* Ponte *under any circumſtance of Cards remaining in the Stock, when he lays his Stake; and of any number of times that his Card is repeated in the Stock.*

The Solution of this Problem containing four caſes, *viz.* of the Ponte's Card being *once*, *twice*, *three* or *four* times in the Stock; we ſhall give the Solution of all theſe caſes ſeverally.

SOLUTION of the *firſt* Caſe.

The Ponte has the following chances to win or loſe, according to the place his Card is in.

1	1 Chance for winning	0
2	1 Chance for loſing	y
3	1 Chance for winning	1
4	1 Chance for loſing	1
5	1 Chance for winning	1
6	1 Chance for loſing	1
*	1 Chance for winning	0

It appears by this Scheme, that he has as many Chances to win 1 as to loſe 1, and that there are two Chances for neither winning or loſing, *viz.* the firſt and the laſt, and therefore that his only Loſs is upon account of his being *Faced:* from which it is plain that the number of Cards covered by that which is in view being called n, his Loſs will be $\frac{y}{n}$, or $\frac{1}{2n}$, ſuppoſing $y=\frac{1}{2}$.

SOLUTION of the *ſecond* Caſe.

By the firſt Remark belonging to the xth Problem, it appears [†] that the Chances which the Ponte has to win or loſe are proportional to the numbers, $n-1$, $n-2$, $n-3$, &c. Wherefore his Chances for winning and loſing may be expreſſed by the following Scheme.

[†] Namely, by calling the Ponte's two Cards two white Counters, drawn for alternately by A and B; and ſuppoſing all A's Chances to belong to the Banker's right hand, and thoſe of B to his left. And the like for the Caſes of the Ponte's Card being in the Stock 3 or 4 times.

1	$n-1$	Chances for winning	0
2	$n-2$	Chances for loſing	y
3	$n-3$	Chances for winning	1
4	$n-4$	Chances for loſing	1
5	$n-5$	Chances for winning	1
6	$n-6$	Chances for loſing	1
7	$n-7$	Chances for winning	1
8	$n-8$	Chances for loſing	1
9	$n-9$	Chances for winning	1
*	1	Chance for loſing	1

Now ſetting aſide the firſt and ſecond number of Chances, it will be found that the difference between the 3d and 4th is $= 1$, that the difference between the 5th and 6th is alſo $= 1$, and that the difference between the 7th and 8th is alſo $= 1$, and ſo on. But the number of differences is $\frac{n-3}{2}$, and the Sum of all the Chances is $\frac{n}{1} \times \frac{n-1}{2}$: wherefore the Gain of the Ponte is $\frac{n-3}{n \times n-1}$. But his Loſs upon account of the *Face* is $\overline{n-2} \times y$ divided by $\frac{n \times n-1}{1 \times 2}$ that is $\frac{\overline{2n-4} \times y}{n \times n-1}$: hence it is to be concluded that his Loſs upon the whole is $\frac{\overline{2n-4} \times y - \overline{n-3}}{n \times n-1}$ or $\frac{1}{n \times n-1}$ ſuppoſing $y = \frac{1}{2}$.

That the number of differences is $\frac{n-3}{2}$ will be made evident from two conſiderations.

Firſt, the Series $n-3$, $n-4$, $n-5$, &c. decreaſes in Arithmetic Progreſſion, the difference of its terms being Unity, and the laſt Term alſo Unity, therefore the number of its Terms is equal to the firſt Term $n-3$: but the number of differences is one half of the number of Terms; therefore the number of differences is $\frac{n-3}{2}$.

Secondly, it appears, by the xth Problem, that the number of all the Terms including the two firſt is always $b+1$, but a in this caſe is $= 2$, therefore the number of all the Terms is $n-1$; from which excluding the two firſt, the number of remaining Terms will be $n-3$, and conſequently the number of differences $\frac{n-3}{2}$.

That the Sum of all the Terms is $\frac{n}{1} \times \frac{n-1}{2}$, is evident alſo from two different conſiderations.

Firſt in any Arithmetic Progreſſion whereof the firſt Term is $n - 1$, the difference Unity, and the laſt Term alſo Unity, the Sum of the Progreſſion will be $\frac{n}{1} \times \frac{n-1}{2}$.

Secondly, the Series $\frac{2}{u \times n-1} \times \overline{n-1} + \overline{n-2} + \overline{n-3}$, &c. mentioned in the firſt Remark upon the tenth Problem, expreſſes the Sum of the Probabilities of winning which belong to the ſeveral Gameſters in the caſe of two white Counters, when the number of all the Counters is n. It therefore expreſſes likewiſe the Sum of the Probabilities of winning which belong to the Ponte and Banker in the preſent caſe: but this Sum muſt always be equal to Unity, it being a certainty that the Ponte or Banker muſt win; ſuppoſing therefore that $n - 1 + n - 2 + n - 3$, &c. is $= S$, we ſhall have the Equation $\frac{2S}{n \times n-1} = 1$, and therefore $S = \frac{n}{1} \times \frac{n-1}{2}$.

SOLUTION of the *third* Caſe.

By the firſt Remark of the tenth Problem, it appears that the Chances which the Ponte has to win and loſe, may be expreſſed by the following Scheme.

1	$n-1 \times n-2$	for winning	0
2	$n-2 \times n-3$	for loſing	y
3	$n-3 \times n-4$	for winning	1
4	$n-4 \times n-5$	for loſing	1
5	$n-5 \times n-6$	for winning	1
6	$n-6 \times n-7$	for loſing	1
7	$n-7 \times n-8$	for winning	1
8	$n-8 \times n-9$	for loſing	1
*	2×1	for winning	1

Setting aſide the firſt, ſecond, and laſt number of Chances, it will be found that the difference between the 3d and 4th is $2n-8$; the difference between the 5th and 6th, $2n-12$; the difference between the 7th and 8th, $2n-16$, &c. Now theſe differences conſtitute an Arithmetic Progreſſion, whereof the firſt Term is $2n-8$, the common difference 4, and the laſt Term 6, being the difference between 4×3 and 3×2. Wherefore the Sum of this Progreſſion is $\frac{n-1}{1} \times \frac{n-5}{2}$, to which adding the laſt Term 2×1, which is favourable to the Ponte, the Sum total will be $\frac{n-3}{1} \times \frac{n-3}{2}$: but the Sum of all the Chances is $\frac{n}{1} \times \frac{n-1}{1} \times \frac{n-2}{3}$, as may be collected from the firſt Remark of the x^{th} Problem, and the laſt Paragraph of the ſecond caſe of this Problem: therefore the Gain of the Ponte is $\frac{3 \cdot n-3 \cdot n-3}{2 \cdot n \cdot n-1 \cdot n-2}$. But his Loſs upon account of the *Face* is $\frac{3 \cdot n-2 \cdot n-3 \cdot y}{n \cdot n-1 \cdot n-2}$ or $\frac{3y \cdot n-3}{n \cdot n-1}$, therefore his Loſs upon the whole is $\frac{3y \cdot n-3}{n \cdot n-1} - \frac{3 \cdot n-3 \cdot n-3}{2 \cdot n \cdot n-1 \cdot n-2}$; or $\frac{3n-9}{2 \cdot n \cdot n-1 \cdot n-2}$ ſuppoſing $y = \frac{1}{2}$.

SOLUTION of the *fourth* Caſe.

The Chances of the Ponte may be expreſſed by the following Scheme.

1	$n-1 \times n-2 \times n-3$	for winning	0
2	$n-2 \times n-3 \times n-4$	for loſing	y
3	$n-3 \times n-4 \times n-5$	for winning	1
4	$n-4 \times n-5 \times n-6$	for loſing	1
5	$n-5 \times n-6 \times n-7$	for winning	1
6	$n-6 \times n-7 \times n-8$	for loſing	1
7	$n-7 \times n-8 \times n-9$	for winning	1
*	$3 \times 2 \times 1$	for loſing	1

Setting aſide the firſt and ſecond numbers of Chances, and taking the differences between the 3^d and 4^{th}, 5^{th} and 6^{th}, 7^{th} and 8^{th}, the laſt of theſe differences will be found to be 18. Now if the number of thoſe differences be p, and we begin from the laſt 18, their Sum, from the ſecond Remark of the x^{th} Problem, will be found to be $p \times \overline{p+1} \times \overline{4p+5}$, but p in this caſe is $= \frac{n-5}{2}$, and therefore the Sum of theſe differences will eaſily appear to be $\frac{n-5}{2} \times \frac{n-3}{2} \times \frac{2n-5}{1}$, but the Sum of all the Chances is $\frac{n}{1} \times \frac{n-1}{1} \times \frac{n-2}{1} \times \frac{n-3}{4}$; wherefore the Gain of the Ponte is $\frac{n-5 \cdot n-3 \cdot 2n-5}{n \cdot n-1 \cdot n-2 \cdot n-3}$: now his Loſs upon account of the Face is $\frac{n-2 \cdot n-3 \cdot n-4 \cdot 4y}{n \cdot n-1 \cdot n-2 \cdot n-3}$, and therefore his Loſs upon the whole will be $\frac{n-4 \cdot 4y}{n \cdot n-1} - \frac{n-5 \cdot 2n-5}{n \cdot n-1 \cdot n-2}$ or $\frac{3n-9}{n \cdot n-1 \cdot n-2}$, ſuppoſing $y = \frac{1}{2}$.

There ſtill remains one ſingle caſe to be conſidered, *viz.* what the Loſs of the Ponte is, when he lays a Stake before the Pack is turned up: but there will be no difficulty in it, after what we have ſaid; the difference between this caſe and the reſt being only, that he is liable to be faced by the firſt Card diſcovered, which will make his Loſs to be $\frac{3n-6}{n\,.\,n-1\,.\,n-3}$, that is, interpreting n by the number of all the Cards in the Pack, *viz.* 52, about $\frac{1}{866}$ part of his Stake.

From what has been ſaid, a Table may eaſily be compoſed, ſhewing the ſeveral Loſſes of the Ponte in whatever circumſtance he may happen to be.

A Table for BASSETTE.

N	1	2	3	4
52	* * *	* * *	* * *	866
51	* * *	* * *	1735	867
49	98	2352	1602	801
47	94	2162	1474	737
45	90	1980	1351	675
43	86	1806	1234	617
41	82	1640	1122	561
39	78	1482	1015	507
37	74	1332	914	457
35	70	1190	818	409
33	66	1056	727	363
31	62	930	642	321
29	58	812	562	281
27	54	702	487	243
25	50	600	418	209
23	46	506	354	177
21	42	420	295	147
19	38	342	242	121
17	34	272	194	97
15	30	210	151	75
13	26	156	114	57
11	22	110	82	41
9	18	72	56	28
7	14	42	35	17

The uſe of this Table will be beſt explained by ſome Examples.

EXAMPLE 1.

Let it be propoſed to find the Loſs of the Ponte, when there are 26 Cards remaining in the Stock, and his Card is twice in it.

In the Column N find the number 25, which is leſs by 1 than the number of Cards remaining in the Stock: over-againſt it, and under the number 2, which is at the head of the ſecond Column, you will find 600; which is the Denominator of a fraction whoſe Numerator is Unity, and which ſhews that his Loſs in that circumſtance is one part in ſix hundred of his Stake.

EXAMPLE 2.

To find the Loſs of the Ponte when there are eight Cards remaining in the Stock, and his Card is three times in it.

In the Column N find the number 7, leſs by one than the number of Cards remaining in the Stock: over-againſt 7, and under the number 3, written on the top of one of the Columns, you will find 35, which denotes that his Loſs is one part in thirty-five of his Stake.

COROLLARY 1.

'Tis plain from the conſtruction of the Table, that the fewer Cards are in the Stock, the greater is the Loſs of the Ponte.

COROLLARY 2.

The leaſt Loſs of the Ponte, under the ſame circumſtances of Cards remaining in the Stock, is when his Card is but twice in it; the next greater but three times; ſtill greater when four times; and the greateſt when but once. If the Loſs upon the Face were varied, 'tis plain that in all the like circumſtances, the Loſs of the Ponte would vary accordingly; but it would be eaſy to compoſe other Tables to anſwer that variation; ſince the quantity y, which has been aſſumed to repreſent that Loſs, having been preſerved in the general expreſſion of the Loſſes, if it be interpreted by $\frac{2}{3}$ inſtead of $\frac{1}{2}$, the Loſs, in that caſe, would be as eaſily determined as in the other: thus ſuppoſing that 8 Cards are remaining in the Stock, and that the Card of the Ponte is twice in it, and alſo that y ſhould be interpreted by $\frac{2}{3}$, the Loſs of the Ponte would be found to be $\frac{4}{63}$ inſtead of $\frac{1}{42}$.

The Game of PHARAON.

The Calculation for *Pharaon* is much like the preceding, the reaſonings about it being the ſame; it will therefore be ſufficient to lay down the Rules of the Play, and the Scheme of Calculation.

Rules of the Play.

Firſt, the Banker holds a Pack of 52 Cards.

Secondly, he draws the Cards one after the other, and lays them down at his right and left-hand alternately.

Thirdly, the Ponte may at his choice ſet one or more Stakes upon one or more Cards, either before the Banker has begun to draw the Cards, or after he has drawn any number of couples.

Fourthly, the Banker wins the Stake of the Ponte, when the Card of the Ponte comes out in an odd place on his right-hand; but loſes as much to the Ponte when it comes out in an even place on his left-hand.

Fifthly, the Banker wins half the Ponte's Stake, when it happens to be twice in one couple.

Sixthly, when the Card of the Ponte being but once in the Stock, happens to be the laſt, the Ponte neither wins nor loſes.

Seventhly, the Card of the Ponte being but twice in the Stock, and the laſt couple containing his Card twice, he then loſes his whole Stake.

PROBLEM XIV.

To find at Pharaon *the Gain of the Banker in any circumſtance of Cards remaining in the Stock, and of the number of times that the Ponte's Cards is contained in it.*

This Problem having four Caſes, that is, when the Ponte's Card is *once*, *twice*, *three*, or *four* times in the Stock; we ſhall give the Solution of theſe four caſes ſeverally.

SOLUTION of the *firſt* Caſe.

The Banker has the following number of Chances for winning and loſing.

1	1 Chance for winning	1
2	1 Chance for loſing	1
3	1 Chance for winning	1
4	1 Chance for loſing	1
5	1 Chance for winning	1
*	1 Chance for loſing	0

Wherefore, the Gain of the Banker is $\frac{1}{n}$, ſuppoſing n to be the number of Cards in the Stock.

SOLUTION of the *ſecond* Caſe.

The Banker has the following Chances for winning and loſing.

1	$n-2$ Chances for winning	1
	1 Chance for winning	y
2	$n-2$ Chances for loſing	1
	$n-4$ Chances for winning	1
3	1 Chance for winning	y
4	$n-4$ Chances for loſing	1
	$n-6$ Chances for winning	1
5	1 Chance for winning	y
6	$n-6$ Chances for loſing	1
	$n-8$ Chances for winning	1
7	1 Chance for winning	y
8	$n-8$ Chances for loſing	1
*	1 Chance for winning	1

The Gain of the Banker is therefore $\frac{\overline{n-2}\cdot y}{n\cdot n-1}+\frac{2}{n\cdot n-1}$, or $\frac{\frac{1}{2}n+1}{n\cdot n-1}$ ſuppoſing $y=\frac{1}{2}$.

The only thing that deſerves to be explained here, is this; how it comes to paſs, that whereas at *Baſſette*, the firſt number of Chances for winning was repreſented by $n-1$, here 'tis repreſented by $n-2$; to anſwer this, it muſt be remembered, that according to the Law of this Play, if the Ponte's Cards come out in an odd place, the Banker is not thereby entitled to the Ponte's whole Stake: for if it ſo happens that his Card comes out again immediately after, the Banker wins but one half of it; therefore the number $n-1$ is divided into two parts, $n-2$ and 1, whereof the firſt is proportional to the Probability which the Banker has for winning the whole Stake of the Ponte, and the ſecond is proportional to the Probability of winning the half of it.

SOLUTION of the *third* Caſe.

The number of Chances which the Banker has for winning and loſing, are as follow:

1	$\begin{cases} n-2 \times n-3 \\ 2 \times n-2 \end{cases}$	Chances for winning Chances for winning	1 y
2	$n-2 \times n-3$	Chances for loſing	1
3	$\begin{cases} n-4 \times n-5 \\ 2 \times n-4 \end{cases}$	Chances for winning Chances for winning	1 y
4	$n-4 \times n-5$	Chances for loſing	1
5	$\begin{cases} n-6 \times n-7 \\ 2 \times n-6 \end{cases}$	Chances for winning Chances for winning	1 y
6	$n-6 \times n-7$	Chances for loſing	1
7	$\begin{cases} n-8 \times n-9 \\ 2 \times n-8 \end{cases}$	Chances for winning Chances for winning	1 y
*	2×1	Chances for loſing	1

Wherefore the Gain of the Banker is $\frac{3y}{2 \cdot n-1}$, or $\frac{3}{4 \cdot n-1}$ ſuppoſing $y = \frac{1}{2}$.

The number of Chances for the Banker to win, is divided into two parts, whereof the firſt expreſſes the number of Chances he has for winning the whole Stake of the Ponte, and the ſecond for winning the half of it.

Now for determining exactly thoſe two parts, it is to be conſidered, that in the firſt couple of Cards that are laid down by the Banker, the number of Chances for the firſt Card to be the Ponte's is $n-1 \times n-2$; alſo, that the number of Chances for the ſecond to be the Ponte's, but not the firſt, is $n-2 \times n-3$: wherefore the number of Chances for the firſt to be the Ponte's, but not the ſecond, is likewiſe $n-2 \times n-3$. Hence it follows, that if from the number of Chances for the firſt Card to be the Ponte's, *viz.* from $n-1 \times n-2$, there be ſubtracted the number of Chances for the firſt to be the Ponte's, and not the ſecond, *viz.* $n-2 \times n-3$, there will remain the number of Chances for both firſt and ſecond Cards to be the Ponte's, *viz.* $2 \times n-2$, and ſo for the reſt.

SOLUTION of the *fourth* Caſe.

The number of Chances which the Banker has for winning and loſing, are as follow:

1	$n-2 \times n-3 \times n-4$	for winning	1
	$3 \times n-2 \times n-3$	for winning	y
2	$n-2 \times n-3 \times n-4$	for loſing	1
	$n-4 \times n-5 \times n-6$	for winning	1
3	$3 \times n-4 \times n-5$	for winning	y
4	$n-4 \times n-5 \times n-6$	for loſing	1
	$n-6 \times n-7 \times n-8$	for winning	1
5	$3 \times n-6 \times n-7$	for winning	y
6	$n-6 \times n-7 \times n-8$	for loſing	1
	$n-8 \times n-9 \times n-10$	for winning	1
7	$3 \times n-8 \times n-9$	for winning	y
8	$n-8 \times n-9 \times n-10$	for loſing	1
	$2 \times 1 \times 0$	for winning	1
*	$3 \times 2 \times 1$	for winning	y
	$2 \times 1 \times 0$	for loſing	1

Wherefore the Gain of the Banker, or the Loſs of the Ponte, is $\frac{2n-5}{n-1 \cdot n-3}y$ or $\frac{2n-5}{2 \times n-1 \cdot n-3}$ ſuppoſing y to be $= \frac{1}{2}$.

It will be eaſy, from the general expreſſions of the Loſſes, to compare the diſadvantage of the Ponte at *Baſſette* and *Pharaon,* under the ſame circumſtances of Cards remaining in the hands of the Banker, and of the number of times that the Ponte's Card is contained in the Stock; but to ſave that trouble, I have thought fit here to annex a Table of the Gain of the Banker, or Loſs of the Ponte, for any particular circumſtance of the Play, as it was done for *Baſſette.*

A Table for PHARAON.

N	1	2	3	4
52	* * *	* * *	* * *	*50
50	* * *	94	65	48
48	48	90	62	46
46	46	86	60	44
44	44	82	57	42
42	42	78	54	40
40	40	74	52	38
38	38	70	49	36
36	36	66	46	34
34	34	62	44	32
32	32	58	41	30
30	30	54	38	28
28	28	52	36	26
26	26	46	33	24
24	24	42	30	22
22	22	38	28	20
20	20	34	25	18
18	18	30	22	16
16	16	26	20	14
14	14	22	17	12
12	12	18	14	10
10	10	14	12	8
8	8	11	9	6

The numbers of the foregoing Table, as well as thoſe of the Table for *Baſſette*, are ſufficiently exact to give at firſt view an idea of the advantage of the Banker in all circumſtances, and the Method of uſing it is the ſame as that which was given for *Baſſette*. It is to be obſerved at this Play, that the leaſt diſadvantage of the Ponte, under the ſame circumſtances of Cards remaining in the Stock, is when the Card of the Ponte is but twice in it, the next greater when three times, the next when once, and the greateſt when four times.

Of PERMUTATIONS *and* COMBINATIONS.

Permutations are the Changes which ſeveral things can receive in the different orders in which they may be placed, being conſidered as taken two and two, three and three, four and four, *&c.*

Combinations are the various Conjunctions which ſeveral things may receive without any reſpect to order, being taken two and two, three and three, four and four.

The Solution of the Problems that relate to Permutations and Combinations depending entirely upon what has been ſaid in the 8th and 9th Articles of the Introduction, if the Reader will be pleaſed to conſult thoſe Articles with attention, he will eaſily apprehend the reaſon of the Steps that are taken in the Solution of thoſe Problems.

PROBLEM XV.

Any number of things a, b, c, d, e, f, *being given, out of which two are taken as it happens: to find the Probability that any of them, as* a, *ſhall be the firſt taken, and any other, as* b, *the ſecond.*

SOLUTION.

The number of Things in this Example being ſix, it follows that the Probability of taking a in the firſt place is $\frac{1}{6}$: let a be conſidered as taken, then the Probability of taking b will be $\frac{1}{5}$; wherefore the Probability of taking a, and then b, is $\frac{1}{6} \times \frac{1}{5} = \frac{1}{30}$:

COROLLARY.

Since the taking *a* in the firſt place, and *b* in the ſecond, is but one ſingle Caſe of thoſe by which ſix Things may change their order, being taken two and two; it follows that the number of Changes or Permutations of ſix Things, taken two and two, muſt be 30.

Univerſally; let *n* be the number of Things; then the Probability of taking *a* in the firſt place, and *b* in the ſecond will be $\frac{1}{n} \times \frac{1}{n-1}$; and the number of Permutations of thoſe Things, taken two and two, will be $n \times \overline{n-1}$.

PROBLEM XVI.

Any number of Things a, b, c, d, e, f, *being given, out of which three are taken as it happens; to find the Probability that* a *ſhall be the firſt taken,* b *the ſecond, and* c *the third.*

SOLUTION.

The Probability of taking *a* in the firſt place is $\frac{1}{6}$: let *a* be conſidered as taken, then the Probability of taking *b* will be $\frac{1}{5}$: ſuppoſe now both *a* and *b* taken, then the Probability of taking *c* will be $\frac{1}{4}$: wherefore the Probability of taking firſt *a*, then *b*, and thirdly *c*, will be $\frac{1}{6} \times \frac{1}{5} \times \frac{1}{4} = \frac{1}{120}$.

COROLLARY.

Since the taking *a* in the firſt place, *b* in the ſecond, and *c* in the third, is but one ſingle Caſe of thoſe by which ſix Things may change their Order, being taken three and three; it follows, that the number of Changes or Permutations of ſix Things taken three and three, muſt be $6 \times 5 \times 4 = 120$.

Univerſally, if *n* be the number of Things; the Probability of taking *a* in the firſt place, *b* in the ſecond, and *c* in the third, will be $\frac{1}{n} \times \frac{1}{n-1} \times \frac{1}{n-2}$; and the number of Permutations of *n* Things taken three and three, will be $n \times \overline{n-1} \times \overline{n-2}$.

GENERAL COROLLARY.

The number of Permutations of n things, out of which as many are taken together as there are Units in p, will be $n \times n - 1 \times n - 2 \times n - 3$, &c. continued to ſo many Terms as there are Units in p.

Thus the number of Permutations of ſix Things taken four and four, will be $6 \times 5 \times 4 \times 3 = 360$, likewiſe the number of Permutations of ſix Things taken all together will be $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

PROBLEM XVII.

To find the Probability that any number of things, whereof ſome are repeated, ſhall all be taken in any order propoſed: for inſtance, that aabbbcccc *ſhall be taken in the order wherein they are written.*

SOLUTION.

The probability of taking a in the firſt place is $\frac{2}{9}$; ſuppoſe one a to be taken, the Probability of taking the other is $\frac{1}{8}$. Let now the two firſt Letters be ſuppoſed taken, the Probability of taking b will be $\frac{3}{7}$: let this be alſo ſuppoſed taken, the Probability of taking another b will be $\frac{2}{6}$: let this be ſuppoſed taken, the Probability of taking the third b will be $\frac{1}{5}$; after which there remaining nothing but the Letter c, the Probability of taking it becomes a certainty, and conſequently is expreſſed by Unity. Wherefore the Probability of taking all thoſe Letters in the order given is $\frac{2}{9} \times \frac{1}{8} \times \frac{3}{7} \times \frac{2}{6} \times \frac{1}{5} \times \frac{1}{1} = \frac{1}{1260}$.

COROLLARY 1.

The number of Permutations which the Letters *aabbbcccc* may receive being taken all together will be $\frac{9 . 8 . 7 . 6 . 5 . 1}{2 . 1 . 3 . 2 . 1 . 1} = 1260$.

COROLLARY 2.

The ſame Letters remaining, the Probability of the Letters being taken in any other given Order will be juſt the ſame as before:

thus the Probability of thoſe Letters being taken in the order *cabacccbb* will be $\frac{1}{1260}$.

GENERAL COROLLARY.

The number of Permutations which any number n of Things may receive being taken all together, whereof the firſt Sort is repeated p times, the ſecond q times, the third r times, the fourth s times, &c. will be the Series $n \times n-1 \times n-2 \times n-3 \times n-4$, &c. continued to ſo many Terms as there are Units in $p+q+r$ or $n-s$ divided by the product of the following Series, *viz.* $p \times p-1 \times p-2$, &c. $q \times q-1 \times q-2$, &c. $r \times r-1 \times r-2$, &c. whereof the firſt muſt be continued to ſo many Terms as there are Units in p, the ſecond to ſo many Terms as there are Units in q, the third to ſo many as there are Units in r, &c.

PROBLEM XVIII.

Any number of Things a, b, c, d, e, f, *being given: to find the Probability that in taking two of them as it may happen, both* a *and* b *ſhall be taken, without any regard to order.*

SOLUTION.

The Probability of taking a or b in the firſt place will be $\frac{2}{6}$; ſuppoſe one of them taken, as for inſtance a, then the Probability of taking b will be $\frac{1}{5}$. Wherefore the Probability of taking both a and b will be $\frac{2}{6} \times \frac{1}{5}$.

COROLLARY.

The taking of both a and b is but one ſingle Caſe of all thoſe by which ſix Things may be combined two and two; wherefore the number of Combinations of ſix Things taken two and two will be $\frac{6}{1} \times \frac{5}{2}$.

Univerſally. The number of Combinations of n Things taken two and two will be $\frac{n}{1} \times \frac{n-1}{2}$.

PROBLEM XIX.

Any number of things a, b, c, d, e, f *being given, to find the Probability that in taking three of them as they happen, they ſhall be any three propoſed, as* a, b, c, *no reſpect being had to order.*

SOLUTION.

The Probability of taking either *a*, or *b*, or *c*, in the firſt place, will be $\frac{3}{6}$; ſuppoſe one of them as *a* to be taken, then the Probability of taking either *b* or *c* in the ſecond place will be $\frac{2}{5}$: again, let either of them be taken, ſuppoſe *b*, then the Probability of taking *c* in the third place will be $\frac{1}{4}$; wherefore the Probability of taking the three things propoſed, *viz.* *a*, *b*, *c*, will be $\frac{3}{6} \times \frac{2}{5} \times \frac{1}{4}$.

COROLLARY.

The taking of *a*, *b*, *c*, is but one ſingle caſe of all thoſe by which ſix Things may be combined three and three; wherefore the number of Combinations of ſix Things taken three and three will be $\frac{6}{1} \times \frac{5}{2} \times \frac{4}{3} = 20$.

Univerſally. The number of Combinations of *n* things combined according to the number *p*, will be the fraction $\frac{n \cdot n-1 \cdot n-2 \cdot n-3 \cdot n-4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$ &c. both Numerator and Denominator being continued to ſo many Terms as there are Units in *p*.

PROBLEM XX.

To find what Probability there is, that in taking at random ſeven Counters out of twelve, whereof four are white and eight black, three of them ſhall be white ones.

SOLUTION.

Firſt, Find the number of Chances for taking three white out of four, which will be $\frac{4}{1} \times \frac{3}{2} \times \frac{2}{3} = 4$.

Secondly, Find the number of Chances for taking four black out of eight: theſe Chances will be found to be $\frac{8}{1} \times \frac{7}{2} \times \frac{6}{3} \times \frac{5}{4} = 70$.

Thirdly, Becauſe every one of the firſt Chances may be joined with every one of the latter, it follows that the number of Chances for taking three white, and four black, will be $4 \times 70 = 280$.

Fourthly, Altho' the caſe of taking four white and three black, be not mentioned in the Problem, yet it is to be underſtood to be implyed in it; for according to the Law of Play, he who does more than he undertakes, is ſtill reputed a winner, unleſs the contrary be expreſly ſtipulated; let therefore the caſe of taking four white out of four be calculated, and it will be found $\frac{4}{1} \times \frac{3}{2} \times \frac{2}{3} \times \frac{1}{4} = 1$

Fifthly, Find the Chances for taking three black Counters out of eight, which will be found to be $\frac{8}{1} \times \frac{7}{2} \times \frac{6}{3} = 56$.

Sixthly, Multiply the two laſt numbers of Chances together, and the Product 56 will denote the number of Chances for taking four white and three black.

And therefore the whole number of Chances, which anſwer to the conditions of the Problem, are $280 + 56 = 336$.

There remains now to find the whole number of Chances for taking ſeven Counters out of twelve, which will be $\frac{12}{1} \times \frac{11}{2} \times \frac{10}{3} \times \frac{9}{4} \times \frac{8}{5} \times \frac{7}{6} \times \frac{6}{7} = 792$.

Laſtly, Divide therefore 336 by 792, and the Quotient $\frac{336}{792}$ or $\frac{14}{33}$ will expreſs the Probability required; and this Fraction being ſubtracted from Unity, the remainder will be $\frac{19}{33}$, and therefore the Odds againſt taking three white Counters are 19 to 14.

COROLLARY.

Let a be the number of white Counters, b the number of black, n the whole number of Counters $= a + b$, c the number of Counters to be taken out of the number n; let alſo p repreſent the number of white Counters to be found preciſely in c, then the number of Chances for taking none of the white, or one ſingle white, or two white and no more, or three white and no more, or four white and no more, &c. will be expreſſed as follows;

$1 \times \frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3} \times \frac{a-3}{4}$, &c. $\times \frac{b}{1} \times \frac{b-1}{2} \times \frac{b-2}{3}$, &c.

The number of Terms in which a enters being equal to the number p, and the number of Terms in which b enters being equal to the number $c-p$.

And the number of all the Chances for taking a certain number c of Counters out of the number n, is expreſſed by the Series $\frac{n}{1} \times \frac{n-1}{2} \cdot \frac{n-2}{3} \times \frac{n-3}{4}$, &c. to be continued to as many Terms as there are Units in c, for a Denominator.

EXAMPLES.

Suppoſe as in the laſt problem; only that of the 7 Counters drawn, there ſhall not be one white. In this Caſe, ſince $p = 0$, and $c - p = 7 = b - 1$: we are to take 1 of the firſt Series, and 7 (or 1) Terms of the ſecond; which gives the number of Chances 1×8; the Ratio of which to all the 7's that can be taken out of 12, is $\frac{8}{792} = \frac{1}{99}$. So that there is the Odds of 98 to 1, that there ſhall be one or more white Counters among the 7 that are drawn.

Again, if there is to be 1 white Counter and no more, we are now to take the Terms $1 \times \frac{a}{1} \ldots \times \frac{b}{1} \times \frac{b-1}{2} \times \frac{b-2}{3} \times \frac{b-3}{4} \times \frac{b-4}{5} \times \frac{b-5}{6} = 4 \ldots \times \frac{8}{1} \times \frac{7}{2} \times \frac{6}{3} \times \frac{5}{4} \times \frac{4}{5} \times \frac{3}{6} = 4 \times \frac{8 \times 7}{2} = \frac{112}{}$: Which gives the probability $\frac{112}{792} = \frac{14}{99}$; or the odds 85 to 14; that there ſhall be more than 1 white Counter, or that all the 7 ſhall be black.

Laſtly, If it is undertaken to draw all the 4 white among the ſeven, the Number of Chances will be $1 \times \frac{8}{1} \times \frac{7}{2} \times \frac{6}{3} = 56$. And the Probability $\frac{56}{792} = \frac{7}{99}$; that is, the Odds of 92 to 7 that there ſhall be, of the 7 drawn, fewer than 4 white Counters, or none at all.

REMARK.

If the numbers n and c were large, ſuch as $n = 40000$ and $c = 8000$, the foregoing Method would ſeem impracticable, by reaſon of the vaſt number of Terms to be taken in both Series, whereof the firſt is to be divided by the ſecond: tho' if thoſe Terms were actually ſet down, a great many of them being common Diviſors might be expunged out of both Series; for which reaſon it will be convenient to uſe the following Theorem, which is a contraction of that Method, and which will be chiefly of uſe when the white Counters are but few.

Let therefore n be the number of all the Counters; a the number of white; c the number of Counters to be taken out of the number n; p the number of the white that are to be taken preciſely in the number c; then making $n - c = d$. The Probability of taking preciſely the number p of white Counters, will be

$$\frac{c \, . \, c-1 \, . \, c-2, \text{ \&c.} \times d \, . \, d-1 \, . \, d-2, \text{ \&c.} \times \frac{a}{1} \times \frac{a-1}{2} \times \frac{a-2}{3}, \text{ \&c.}}{n \, . \, n-1 \, . \, n-2 \, . \, n-3 \, . \, n-4 \, . \, n-5 \, . \, n-6 \, . \, n-7 \, . \, n-8, \text{ \&c.}}$$

Here it is to be obſerved, that the Numerator conſiſts of three Series which are to be multiplied together; whereof the firſt contains as many Terms as there are Units in p; the ſecond as many as there are Units in $a-p$; the third as many as there are Units in p; and the Denominator as many as there are Units in a.

PROBLEM XXI.

In a Lottery conſiſting of 40000 *Tickets, among which are three particular Benefits, what is the Probability that taking* 8000 *of them, one or more of the particular Benefits ſhall be amongſt them.*

SOLUTION.

Firſt, In the Theorem belonging to the Remark of the foregoing Problem, having ſubſtituted reſpectively 8000, 40000, 32000, 3 and 1, in the room of c, n, d, a, and p; it will appear that the Probability of taking one preciſely of the three particular Benefits, will be $\frac{8000 \, . \, 32000 \, . \, 31999 \, . \, 3}{40000 \, . \, 39999 \, . \, 39998} = \frac{48}{125}$ nearly.

Secondly, c, n, d, a being interpreted as before, let us ſuppoſe $p = 2$: hence the Probability of taking preciſely two of the particular Benefits will be found to be $\frac{8000 \quad 7999 \, . \, 32000 \, . \, 3}{40000 \, . \, 39999 \, . \, 39998} = \frac{12}{125}$ nearly.

Thirdly, making $p = 3$, the Probability of taking all the three particular Benefits will be found to be $\frac{8000 \,.\, 7999 \,.\, 7998}{40000 \,.\, 39999 \,.\, 39998} = \frac{1}{125}$.

Wherefore the Probability of taking one or more of the three particular Benefits will be $\frac{48+12+1}{125}$ or $\frac{61}{125}$ very near.

It is to be obſerved, that thoſe three Operations might have been contracted into one, by inquiring the Probability of not taking any of the three particular Benefits, which will be found to be $\frac{32000 \,.\, 31999 \,.\, 31998}{40000 \,.\, 39999 \,.\, 39998} = \frac{64}{125}$ nearly, which being ſubtracted from Unity, the remainder $1 - \frac{64}{125}$ or $\frac{61}{125}$ will ſhew the Probability required, and therefore the Odds againſt taking any of three particular Benefits will be 64 to 61 nearly.

PROBLEM XXII.

To find how many Tickets ought to be taken in a Lottery conſiſting of 40000, *among which are Three particular Benefits, to make it as probable that one or more of thoſe Three may be taken as not.*

SOLUTION.

Let the number of Tickets requiſite to be taken be $= x$; it will follow therefore from the Remark belonging to the xxth Problem, that the Probability of not taking any of the particular Benefits will be $\frac{n-x}{n} \times \frac{n-x-1}{n-1} \times \frac{n-x-2}{n-2}$; but this Probability is equal to $\frac{1}{2}$, ſince by Hypotheſis the Probability of taking one or more of them is equal to $\frac{1}{2}$, from whence we ſhall have the Equation $\frac{n-x}{n} \times \frac{n-x-1}{n-1} \times \frac{n-x-2}{n-2} = \frac{1}{2}$, which Equation being ſolved, the Value of x will be found to be nearly 8252.

N. B. The Factors whereof both the Numerator and Denominator are compoſed, being but few, and in arithmetic progreſſion; and beſides, the difference being very ſmall in reſpect of n; thoſe Terms may be conſidered as being in geometric Progreſſion: wherefore the Cube of the middle Term $\frac{n-x-1}{n-1}$, may be ſuppoſed equal to the product of the Multiplication of thoſe Terms; from whence will ariſe the Equation $\frac{\overline{n-x-1}|^3}{\overline{n-1}|^3} = \frac{1}{2}$; or, neglecting Unity in both Numerator and Denominator, $\frac{\overline{n-x}|^3}{n^3} = \frac{1}{2}$ and conſequently x will be found to be $= n \times \overline{1 - \sqrt[3]{\frac{1}{2}}}$ or $n \times \overline{1 - \frac{1}{2}\sqrt[3]{4}}$, but $n = 40000$, and $1 - \frac{1}{2}\sqrt[3]{4} = 0.2063$; wherefore $x = 8252$.

In the Remark belonging to the ſecond Problem, a Rule was given for finding the number of Tickets that were to be taken to make it as probable, that one or more of the Benefits would be taken as not; but in that Rule it was ſuppoſed, that the proportion of the Blanks to the Prizes was often repeated, as it uſually is in Lotteries: now in the caſe of the preſent Problem, the particular Benefits being but three in all, the remaining Tickets are to be conſidered as Blanks in reſpect of them; from whence it follows, that the proportion of the number of Blanks to one Prize being very near as 13332 to 1, and that proportion being repeated but three times in the whole number of Tickets, the Rule there given would not have been ſufficiently exact, for which reaſon it was thought neceſſary to give another Rule in this place.

PROBLEM XXIII.

Suppoſing a Lottery of 100000 *Tickets, whereof* 90000 *are Blanks, and* 10000 *are Benefits, to determine accurately what the odds are of taking or not taking a Benefit, in any number of Tickets aſſigned.*

SOLUTION.

Suppoſe the number of Tickets to be 6; then let us inquire into the Probability of taking no Prize in 6 Tickets, which to find let us make uſe of the Theorem ſet down in the Corollary of the xx$^{\text{th}}$ *Problem*, wherein it will appear that the number of Chances for taking no Prize in 6 Tickets, making $a = 10000$, $b = 90000$, $c = 6$, $p = 0$, $n = 100000$, will be

$$\frac{90000}{1} \times \frac{89999}{2} \times \frac{89998}{3} \times \frac{89997}{4} \times \frac{89996}{5} \times \frac{89995}{6},$$

and that the whole number of Chances will be

$$\frac{100000}{1} \times \frac{99999}{2} \times \frac{99998}{3} \times \frac{99997}{4} \times \frac{99996}{5} \times \frac{99995}{6};$$ then

dividing the firſt number of Chances by the ſecond, which may eaſily be done by Logarithms, the Quotient will be 0.53143, and this ſhews the Probability of taking no Prize in 6 Tickets: and this decimal fraction being ſubtracted from Unity, the Remainder 0.46857 ſhews the Probability of taking one Prize or more in 6 Tickets; wherefore the Odds againſt taking any Prize in 6 Tickets, will be 53143 to 46857.

If we ſuppoſe now that the number of Tickets taken is 7, then carrying each number of Chances above-written one ſtep farther, we ſhall find that the Probability of taking no Prize in 7 Tickets is 0.47828, which fraction being ſubtracted from Unity, the remainder will be 0.52172, which ſhews the Odds of taking one Prize or more in 7 Tickets to be 52172 to 47828.

REMARK.

When the number of Tickets taken bear a very inconſiderable proportion to the whole number of Tickets, as it happens in the caſe of this Problem, the Queſtion may be reſolved as a Problem depending on the Caſt of a Die: we may therefore ſuppoſe a Die of 10 Faces having one of its Faces ſuch as the Ace repreſenting a Benefit, and all the other nine repreſenting Blanks, and inquire into the Probability of miſſing the Ace 6 times together, which by the Rules given in the Introduction, will be found to be $\frac{9^6}{10^6} = 0.53144$ differing from what we had found before but one Unit in the fifth place of Decimals. And if we inquire into the Probability of miſſing the Ace 7 times, we ſhall find it 0.47829 differing alſo but one Unit in the fifth of Decimals, from what had been found before, and therefore in ſuch caſes as this we may uſe both Methods indifferently; but the firſt will be exact if we actually multiply the numbers together, the ſecond is only an approximation.

But both Methods confirm the truth of the practical Rule given in our third Problem, about finding what number of Tickets are neceſſary for the equal Chance of a Prize; for multiplying as it is there directed, the number 9 repreſenting the Blanks by 0.7, the Product 6.3 will ſhew that the number requiſite is between 6 and 7.

PROBLEM XXIV.

The ſame things being given as in the preceding Problem, ſuppoſe the price of each Ticket to be 10 L. *and that after the Lottery is drawn,* 7 L. — 10 ſh. *be returned to* *to the Blanks, to find in this Lottery the value of the Chance of a Prize.*

SOLUTION.

There being 90000 Blanks, to every one of which 7 *L.* — 10 *ſh.* is returned, the total Value of the Blanks is 675000 *L.* and conſequently the total Value of the Benefits is 325000 *L.* which being divided by 10000, the number of the Benefits, the Quotient is 32 *L.* — 10 *ſh.*; and therefore one might for the Sum of 32 *L.* — 10 *ſh.* be intitled to have a Benefit certain, taken at random out of the whole number of Benefits: the Purchaſer of a Chance has therefore 1 Chance in 10 for the Sum of 32 *L.* — 10 *ſh.* and 9 Chances in 10 for loſing his Money; from whence it follows, that the value of his Chance is the 10th part of 32 *L.* — 10 *ſh.* that is 3 *L.* — 5 *ſh.* And therefore the Purchaſer of a Chance, by giving the Seller 3 *L.* — 5 *ſh.* is intitled to the Chance of a Benefit, and ought not to return any thing to the Seller, altho' he ſhould have a Prize; for the Seller having 3 *L.* — 5 *ſh.* ſure, and 9 Chances in 10 for 7 *L.* — 10 *ſh.* the Value of which Chances is 6 *L.* — 15 *ſh.*; it follows that he has his 10 *L.*

PROBLEM XXV.

Suppoſing ſtill the ſame Lottery as has been mentioned in the two preceding Problems, let A *engage to furniſh* B *with a Chance, on condition that whenever the Ticket on which the Chance depends, ſhall happen to be drawn, whether it proves a Blank or a Prize,* A *ſhall furniſh* B *with a new Chance, and ſo on, as often as there is occaſion, till the whole Lottery be drawn; to find what conſideration* B *ought to give* A *before the Lottery begins to be drawn, for the Chance or Chances of one or more Prizes, admitting that the Lottery will be* 40 *days a drawing.*

SOLUTION.

Let 3 *L.* — 5 *ſh.*, which is the abſolute Value of a Chance, be called s.

1°. *A* who is the Seller ought to conſider, that the firſt Day, he furniſhes neceſſarily a Chance whoſe Value is s.

2°. That the ſecond day, he does not neceſſarily furniſh a Chance, but conditionally, *viz.* if it ſo happen that the Ticket on which the Chance depends, ſhould be drawn on the firſt day; but the Probability of its being drawn on the firſt day is $\frac{1}{40}$; and therefore he ought to take $\frac{1}{40}s$ for the conſideration of the ſecond day.

3°. That in the ſame manner, he does not neceſſarily furniſh a Chance on the third day, but conditionally, in caſe the only Ticket depending (for there can be but one) ſhould happen to be drawn on the ſecond day; of which the Probability being $\frac{1}{39}$, by reaſon of the remaining 39 days from the ſecond incluſive to the laſt, it follows, that the Value of that Chance is $\frac{1}{39}s$.

4°. And for the ſame reaſon, the Value of the next is $\frac{1}{38}s$, and ſo on.

The Purchaſer ought therefore to give the Seller

$1 + \frac{1}{40} + \frac{1}{39} + \frac{1}{38} + \frac{1}{37} \ldots\ldots\ldots + \frac{1}{2}$, the whole multiplied by s, or

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \ldots\ldots + \frac{1}{40}$, the whole multiplied by s. Now it being pretty laborious to ſum up thoſe 40 Terms, I have here made uſe of a Rule which I have given in the Supplement to my *Miſcellanea Analytica* *, whereby one may in a very ſhort time ſum up as many of thoſe Terms as one pleaſes, tho' they were 10000 or more; and by that Rule, the Sum of thoſe 40 Terms will be found to be 4.2785 very near, which being multiplied by s which in this caſe is 3.25, the product 13.9 will ſhew that the Purchaſer ought to give the Seller about 13 *L.* — 18 *ſh.*

COROLLRAY.

The Value of the Chance s for one ſingle day that ſhall be fixed upon, is the Value of that Chance divided by the number of Days intercepted between that Day incluſive, and the number of Days remaining to the end of the Lottery: which however muſt be underſtood with this reſtriction, that the Day fixed upon muſt be choſe before the Lottery begins; or if it be done on any other Day, the State of the Lottery muſt be known, and a new Calculation made accordingly for the Value of *s*.

* SCHOLIUM.

If there is a Series of Fractions of this Form $\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\frac{1}{n+4}\ldots\ldots+\frac{1}{a-1}$; the firſt of which is $\frac{1}{n}$, and the laſt $\frac{1}{a-1}$; their Sum will be,

$$\left.\begin{array}{l} log.\ \frac{a}{n}+\frac{1}{2n}+\frac{1}{2n^2}A+\frac{1}{4n^4}B+\frac{1}{6n^6}C+\frac{1}{8n^8}D+\&c. \\ \qquad -\frac{1}{2a}+\frac{1}{2a^2}A+\frac{1}{4a^4}B+\frac{1}{6a^6}C+\frac{1}{8a^8}D+\&c. \end{array}\right.$$

Where it is to be obſerved,

1°. That the mark (*log.*) denoting *Neper's*, or the *Natural*, Logarithm, affects only the firſt Term $\frac{a}{n}$.

2°. That the Values of the Capital Letters are, $A=\frac{1}{6}$, $B=-\frac{1}{30}$ $C=+\frac{1}{42}$, $D=-\frac{1}{30}$, $E=+\frac{5}{66}$, &c. being the numbers of Mr. *James Bernoulli* in his excellent Theorem for the Summing of Powers; which are formed from each other as follows;

$$A=\frac{1}{2}-\frac{1}{3}$$
$$B=\frac{1}{2}-\frac{1}{5}-\frac{4}{2}A.$$
$$C=\frac{1}{2}-\frac{1}{7}-\frac{6}{2}A-\frac{6\cdot5\cdot4}{2\cdot3\cdot4}B.$$
$$D=\frac{1}{2}-\frac{1}{9}-\frac{8}{2}A-\frac{8\cdot7\cdot6}{2\cdot3\cdot4}B-\frac{8\cdot7\cdot6\cdot5\cdot4}{2\cdot3\cdot4\cdot5\cdot6}C.$$
$$E=\frac{1}{2}-\frac{1}{11}-\frac{10}{2}A-\frac{10\cdot9\cdot8}{2\cdot3\cdot4}B-\frac{10\cdot9\cdot8\cdot7\cdot6}{2\cdot3\cdot4\cdot5\cdot6}C-\frac{10\cdot9\cdot8\cdot7\cdot6\cdot5\cdot4}{2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}D.$$

&c.

3°. In working by this Rule, it will be convenient to ſum a few of the firſt terms, in the common way; that the powers of $\frac{1}{n}$ may the ſooner converge.

4°. The ſame Rule furniſhes an eaſy Computation of the Logarithm of any ratio $\frac{a}{n}$, the difference of whoſe terms is not very great.

PRO-

PROBLEM XXVI.

To find the Probability of taking four Hearts, three Diamonds, two Spades, and one Club in ten Cards out of a Stock containing thirty-two.

SOLUTION.

Firſt, The number of Chances for taking four Hearts out of the whole number of Hearts that are in the Stock, that is out of Eight, will be $\frac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4}=70$.

Secondly, The number of Chances for taking three Diamonds out of Eight, will be $\frac{8\cdot7\cdot6}{1\cdot2\cdot3}=56$.

Thirdly, The number of Chances for taking two Spades out of Eight, will be $\frac{8\cdot7}{1\cdot2}=28$.

Fourthly, The number of Chances for taking one Club out of Eight, will be $\frac{8}{1}=8$.

And therefore multiplying all thoſe particular Chances together, the product $70\times56\times28\times8=878080$ will denote the whole number of Chances for taking four Hearts, three Diamonds, two Spades, and one Club.

Fifthly, The whole number of Chances for taking any ten Cards out of thirty-two is

$$\frac{32\cdot31\cdot30\cdot29\cdot28\cdot27\cdot26\cdot25\cdot24\cdot23}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10}=64512240.$$

And therefore dividing the firſt Product by the ſecond, the quotient $\frac{878080}{64512240}$ or $\frac{1}{75}$ nearly, will expreſs the Probability required; from which it follows that the Odds againſt taking four Hearts, three Diamonds, two Spades, and one Club in ten Cards, out of a Stock containing thirty-two, are very near 74 to 1.

REMARK.

But if the numbers in this Problem had not been reſtricted each to a particular ſuit of Cards; that is, if it had been undertaken only that in drawing the ten Cards, 4 of them ſhould be of one ſuit, 3 of another, 2 of another, and one of the fourth; then writing for the four ſuits, the Letters $A\ .\ B\ .\ C\ .\ D$; and under them the Numbers $4\cdot3\cdot2\ .\ 1$; ſince this is but one Poſition out of 24, which the numbers can have with reſpect to the Letters (by the general Corollary to Prob. xvi) we muſt now multiply the number of Chances before found, which was 878080, by 24; and the probability required will be $\frac{2107392}{6451224}$; that is, it is the Odds of about 2 to 1, or very nearly of 68 to 33, that of 10 Cards drawn out of a *Piquet* pack *four*, *three*, *two*, and *one*, ſhall not be of different ſuits.

Of the Game of QUADRILLE.

PROBLEM XXVII.

The Player having 3 Matadors and three other Trumps by the loweſt Cards in black or red, what is the Probability of his forcing all the Trumps?

SOLUTION.

In order to ſolve this Problem, it is to be conſidered, that the Player whom I call *A* forces the Trumps neceſſarily, if none of the other Players whom I call *B*, *C*, *D*, has more than three Trumps; and therefore, if we calculate the Probability of any one of them having more than three Trumps, which caſe is wholly againſt *A*, we may from thence deduce what will be favourable to him; but let us firſt ſuppoſe that he plays in black.

Since the number of Trumps in black is 11, and that *A* by ſuppoſition has 6 of them, then the number of Trumps remaining amongſt *B*, *C*, *D* is 5; and again, ſince the number of all the other remaining Cards, which we may call Blanks, is 29, whereof *A* has 4, it follows that there are 25 Blanks amongſt *B*, *C*, *D*; and therefore the number of Chances for *B* in his 10 Cards to have 4 Trumps and 6 Blanks, is by the Corollary of the xx^th^ Problem.

$$\frac{5 \times 4 \times 3 \times 2}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$

And likewiſe the number of Chances for his having 5 Trumps and 5 Blanks, is by the ſame Corollary.

$$\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \quad 3 \cdot 4 \cdot 5} \times \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

O And

And therefore the number of all the Chances of *B* againſt *A* is $106 \times 5 \times 7 \times 11 \times 23$: but the number of Chances whereby any 10 Cards may be taken out of 30 is $\frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}$. which being reduced to $5 \times 7 \times 9 \times 11 \times 13 \times 23 \times 29$, it follows that the Probability of *B*'s having more than three Trumps is $\frac{106 \times 5 \cdot 7 \cdot 11 \cdot 23}{3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 23 \cdot 29} = \frac{106}{9 \times 13 \cdot 29}$: but this Probability falls as well upon *C* and *D* as upon *B*, and therefore it ought to be multiplied by 3, which will make it $\frac{106}{3 \times 13 \times 29} = \frac{106}{1131}$; and this being ſubtracted from Unity, the remainder $\frac{1025}{1131}$ will expreſs the Probability of *A*'s forcing all the Trumps; and therefore the Odds of his forcing the Trumps are 1025 againſt 106, that is 29 to 3 nearly.

But if *A* plays the ſame Game in red, his advantage will be conſiderably leſs than before; for there being 12 Trumps in red, whereof he has 6, *B* may have 4, or 5, or 6 of them, ſo that the number of the Chances which *B* has for more than three Trumps will be reſpectively as follows:

$$\frac{6 \times 5 \times 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$
$$\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \times \frac{24 \cdot 23 \cdot 22 \cdot 21 \cdot 20}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$
$$\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \times \frac{24 \cdot 23 \cdot 22 \cdot 21}{1 \cdot 2 \cdot 3 \cdot 4}$$

Now the Sum of all thoſe Chances being $215 \times 23 \times 22 \times 21$, and the Sum of all the Chances for taking any 10 Cards out of 30, being $5 \times 7 \times 9 \times 11 \times 13 \times 23 \times 29$, as appears by the preceding caſe, it follows, that the Probability of *B*'s having more than three Trumps is $\frac{215 \cdot 23 \cdot 22 \cdot 21}{5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 23 \cdot 29} = \frac{86}{3 \cdot 13 \cdot 29}$; but this Probability falling as well upon *C* and *D*, as upon *B*, ought to be multiplied by 3, which will make it $\frac{86}{13 \cdot 29} = \frac{86}{377}$; and this being ſubtracted from Unity, the remainder $\frac{291}{377}$, will expreſs the Probability of *A*'s forcing all the Trumps; and therefore the Odds of his forcing all the Trumps is in this caſe 291 to 86, that is nearly 10 to 3.

PROBLEM XXVIII.

The Player A having Spadille, Manille, King, Queen, and two ſmall Trumps in black, to find the Probability of his forcing all the Trumps.

SOLUTION.

A forces the Trumps neceſſarily, if *Baſte* accompanied with two other Trumps be not in one of the Hands of *B*, *C*, *D*, and as *Baſte* ought to be in ſome Hand, it is indifferent where we place it; let it therefore be ſuppoſed that *B* has it, in conſequence of which let us conſider the number of Chances for his having beſides *Baſte*,

1°. 2 Trumps and 7 Blanks.
2°. 3 Trumps and 6 Blanks.
3°. 4 Trumps and 5 Blanks.

Now the Blanks being in all 29, whereof *A* has 4, it follows that the number of remaining Blanks is 25; and the number of Trumps being in all 11, whereof *A* has 6 by Hypotheſis, and *B* has 1, *viz. Baſte*, it follows that the number of remaining Trumps is 4; and therefore the Chances which *B* has againſt the Player are reſpectively as follows:

$$\frac{4 \cdot 3}{1 \cdot 2} \times \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}$$

$$\frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} \times \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$$

$$\frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

The Sum of all which is $1441 \times 5 \times 23 \times 22$; but the Sum of all the Chances whereby *B* may join any 9 Cards to the *Baſte* which he has already is $\frac{29 \times 28 \times 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \times 22 \cdot 21}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} = 29 \times 7 \times 3 \times 13 \times 5 \times 23 \times 11$. and therefore the Probability of *Baſte* being in one Hand, accompanied with two Trumps at leaſt, is expreſſed by the Fraction $\frac{1441 \cdot 5 \cdot 23 \cdot 22}{29 \cdot 7 \cdot 3 \cdot 13 \cdot 5 \cdot 23 \cdot 11} = \frac{131 \cdot 22}{29 \cdot 7 \cdot 3 \cdot 13} = \frac{2882}{7917}$ and this being ſubtracted from Unity, the remainder will be $\frac{5035}{7917}$, and therefore the Odds of *A*'s forcing the Trumps are 5035 to 2882, which are very near 7 to 4.

But if it be in red, *A* has the ſmall diſadvantage of 19703 againſt 19882, or nearly 110 againſt 111.

It is to be noted in this Propoſition, that it is not now neceſſary to multiply by 3; by reaſon that *B* repreſents indeterminately any one of the three *B, C, D*: elſe if the caſe of having *Baſte* was determined to *B* in particular, his probability of having it would only be $\frac{1}{3}$: ſo that the Chances afterwards being multiplied by 3, the Solution would be the ſame.

PROBLEM XXIX.

The Player having Spadille, Manille, and 5 other Trumps more by the loweſt in red, what is the Probability, by playing Spadille and Manille, of his forcing 4 Trumps?

SOLUTION.

The 5 remaining Trumps being between *B, C, D*, their various diſpoſitions are the following:

B,	*C*,	*D*
1,	2,	2
2,	3,	0
3,	1,	1
4,	1,	0
5,	0,	0

Which muſt be underſtood in ſuch manner, that what is here aſſigned to *B* may as well belong to *C* or *D*.

Now it is plain, that out of thoſe five diſpoſitions there are only the two firſt that are favourable to *A*; let us therefore ſee what is the Probability of the firſt diſpoſition.

The number of Chances of *B* to have 1 Trump and 9 Blanks are $\frac{5}{1} \times \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 \cdot 18 \cdot 17}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} = 5 \times 5 \times 5 \times 11 \times 17 \times 19 \times 23$, but the number of all the Chances whereby he may take any 10 Cards out of 30, is $5 \times 7 \times 9 \times 11 \times 13 \times 23 \times 29$ as has been ſeen already in one of the preceding Problems; and therefore the Probability of *B*'s having one Trump and nine Blanks is $\frac{5 \cdot 5 \cdot 5 \cdot 11 \cdot 17 \cdot 19 \cdot 23}{5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 23 \cdot 29} = \frac{25 \cdot 19 \cdot 17}{7 \cdot 9 \cdot 13 \cdot 29}$.

Now in order to find the number of Chances for *C* to have 2 Trumps and 8 Blanks, it muſt be conſidered that *A* having 7 Trumps, and *B* 1, the number of remaining Trumps is 4; and likewiſe that *A* having 3 Blanks, and *B* 9, the number of remaining Blanks is 16, and therefore that the number of Chances for *C* to have 2 Trumps and 8 Blanks is

$$\frac{4 \cdot 3}{1 \cdot 2} \times \frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} = 6 \times 9 \times 10 \times 11 \times 13.$$

But the number of all the Chances whereby *C* may take any 10 Cards out of 20 remaining between him and *D*, is

$$\frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} = 4 \times 11 \times 13 \times 17 \times 19,$$

and therefore the Probability of *C*'s having 2 Trumps and 8 Blanks is

$$\frac{6 \cdot 9 \cdot 10 \cdot 11 \cdot 13}{4 \cdot 11 \cdot 13 \cdot 17 \cdot 19} = \frac{6 \cdot 9 \cdot 10}{4 \cdot 17 \cdot 19} = \frac{9 \cdot 15}{17 \cdot 19}$$

Now *A* being ſuppoſed to have had 7 Trumps, *B* 1, and *C* 2, *D* muſt have 2 neceſſarily, and therefore no new Calculation ought to be made on account of *D*. It follows therefore that the Probability of the diſpoſition 1, 2, 2, belonging reſpectively to *B*, *C*, *D*, ought to be expreſſed by $\frac{25 \cdot 10 \cdot 17}{7 \cdot 9 \cdot 13 \cdot 29} \times \frac{9 \cdot 15}{17 \cdot 19} = \frac{15 \cdot 25}{7 \cdot 13 \cdot 29}$.

Now three things, whereof two are alike, being to be permuted 3 different ways, it follows that the Probability of the Diſpoſition 1, 2, 2, as it may happen in any order, will be $\frac{3 \cdot 15 \cdot 25}{7 \cdot 13 \cdot 29} = \frac{1125}{2639}$.

It will be found in the ſame manner, that the Probability of the Diſpoſition 2, 3, 0 as it belongs reſpectively to *B*, *C*, *D*, is $\frac{2 \cdot 5 \cdot 10}{7 \cdot 13 \cdot 29}$; but the number of Permutations of three things which are all unlike being 6, it follows that the ſaid Probability ought to be multiplied by 6, which will make it $\frac{6 \cdot 2 \cdot 5 \; 10}{7 \; 13 \cdot 29} = \frac{600}{2639}$.

From all which it follows, that the Probability of *A*'s forcing 4 Trumps is $\frac{1125+600}{26\;9} = \frac{1725}{2639}$; which fraction being ſubtracted from Unity, the remainder will be $\frac{914}{2\;39}$, and therefore the Odds of *A*'s forcing 4 Trumps are 1725 to 914, that is very near 17 to 9.

PROBLEM XXX.

A *the Player having* 4 *Matadors, in Diamonds, with the two black Kings, each accompanied with two ſmall Cards of their own ſuit; what is the Probability that no one of the others* B, C, D, *has more than* 4 *Trumps, or in caſe he has more, that he has alſo of the ſuit of both his Kings; in which caſes* A *wins neceſſarily?*

SOLUTION.

The Chances that are againſt *A* are as follows; it being poſſible that *B* may have

Diamonds,	Hearts,	Number of Chances.
5,	5	14112
6,	4	5880
7,	3	960
8,	2	45
		Sum 20997

Diamonds	Spades,	Hearts,	Number of Chances.
5,	1	4	70560
5,	2	3	100800
5,	3	2	50400
5,	4	1	8400
5,	5	0	336
6,	1	3	20160
6,	2	2	18900
6,	3	1	5600
6,	4	0	420
7,	1	2	2160
7,	2	1	1200
7,	3	0	160
8,	1	1	60
8,	2	0	15
			279171

Now by reaſon that among *B*, *C*, *D*, there are as many Clubs as Spades, *viz.* 6 of each ſort, it follows that the Clubs may be ſubſtituted in the room of the Spades, which will double this laſt number of Chances, and make it 558342; and therefore, adding together the firſt and ſecond number of Chances, the Sum will be 579339, which will be the whole number of Chances, whereby *B* may withſtand the Expectation of *A*; but the number of all the Chances which *B* has for taking any 10 Cards out of 30, is $5 \times 7 \times 9 \times 11 \times 13 \times 23 \times 29 = 30045015$; from which it follows that the Probability of *B*'s withſtanding the Expectation of *A* is $\frac{579339}{30045015}$: but as this may fall as well upon *C* and *D* as upon *B*, it follows that this Probability ought to be multiplied by 3, then the Product $\frac{1738017}{30045015}$ will expreſs the Probability of *A*'s loſing; and this being ſubtracted from Unity, the remainder will expreſs the Probability of *A*'s winning; and therefore the Odds of *A*'s winning will be little more than $16\frac{1}{4}$ to 1.

PROBLEM XXXI.

A *having Spadille, Manille, King, Knave, and two other ſmall Trumps in black, what is the Probability that Baſte accompanied with two other Trumps, or the Queen accompanied with three other, ſhall not be in the ſame hand; in which caſe* A *wins neceſſarily?*

SOLUTION.

The Probability of Baſte being in one hand, accompanied with two other Trumps, has been found, in Problem xxviii, to be $\frac{2882}{7917}$.

The number of Chances for him who has the Queen, to have alſo three other Trumps, excluding Baſte, is

$$\frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3} \times \frac{25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = 5 \times 7 \times 11 \times 20 \times 23$$

but the number of Chances for joining any 9 Cards to the Queen is $3 \times 5 \times 7 \times 11 \times 13 \times 23 \times 29$, and therefore the Probability of the Queen's being in one hand, accompanied with three other Trumps, is

$$\frac{5 \cdot 7 \cdot 11 \cdot 20 \cdot 23}{3 \times 5 \cdot 7 \cdot 11 \cdot 13 \cdot 23 \cdot 29} = \frac{20}{3 \cdot 13 \cdot 29} = \frac{20}{1131} = \frac{140}{7917},$$

now this Probability being added to the former, the Sum will be $\frac{3022}{7917}$; and therefore the Odds of *A*'s not being withſtood either from Baſte being accompanied with two other Trumps, or from the Queen accompanied with three, are 4895 to 3022, nearly as 13 to 8.

It may be obſerved, that the reaſon of Baſte being excluded from among the Trumps that accompany the Queen is this; if the Queen be accompanied with Baſte and two other Trumps, the Baſte itſelf is accompanied with three Trumps, which caſe had been taken in already in the firſt part of the Solution.

PROBLEM XXXII.

A *having three Matadors in Spades with the Kings of Hearts, Diamonds, and Clubs, two ſmall Hearts, and two ſmall Diamonds; to find the Probability that not above three Spades ſhall be in one hand, or that, if there be above three, there ſhall be alſo of the ſuits of the three Kings; in which caſe* A *wins neceſſarily.*

SOLUTION.

The Probability of not above three Trumps being in one hand $=0.332141$.

The Probability that one of the Oppoſers ſhall have 4 Trumps, and at the ſame time Hearts, Diamonds, and Clubs, and that no other ſhall have 4 Trumps, is $=0.393501$.

The Probability that two of the Oppoſers ſhall have 4 Trumps, and at the ſame time Hearts, Diamonds, and Clubs, is $=0.013836$.

The Probability that one of the Oppoſers ſhall have 5 Trumps, and at the ſame time Hearts, Diamonds, and Clubs, is $=0.103019$.

The Probability that one of the Oppoſers ſhall have 6 Trumps, Hearts, Diamonds, and Clubs, is $=0.001041$.

The Probability of one of the Oppoſers having 7 Trumps, and at the ſame time Hearts, Diamonds, and Clubs, is $=0.000313$.

Now the Sum of all theſe Probabilities is 0.843851, which being ſubtracted from Unity, the remainder is 0.156149; and therefore the Odds of the Player's winning are as 843851 to 156149, that is very near as 27 to 5.

PRO-

PROBLEM XXXIII.

To find at Pharaon, *how much it is that the Banker gets* per Cent. *of all the Money that is adventured.*

HYPOTHESIS.

I ſuppoſe *firſt*, that there is but one ſingle Ponte; *Secondly*, that he lays his Money upon one ſingle Card at a time; *Thirdly*, that he begins to take a Card in the beginning of the Game; *Fourthly*, that he continues to take a new Card after the laying down of every couple: *Fifthly*, that when there remain but ſix Cards in the Stock, he ceaſes to take a Card.

SOLUTION.

When at any time the Ponte lays a new Stake upon a Card taken as it happens out of his Book, let the number of Cards already laid down by the Banker be ſuppoſed equal to x, and the whole number of Cards equal to n.

Now in this circumſtance the Card taken by the Ponte has paſt four times, or three times, or twice, or once, or not at all.

Firſt, If it has paſſed four times, he can be no loſer upon that account.

Secondly, If it has paſſed three times, then his Card is once in the Stock: now the number of Cards remaining in the Stock being $n - x$, it follows by the firſt caſe of the xiii$^{\text{th}}$ Problem, that the Loſs of the Ponte will be $\frac{1}{n-x}$: but by the Remark belonging to the xx$^{\text{th}}$ Problem, the Probability of his Card's having paſſed three times preciſely in x Cards is $\frac{x \cdot x-1 \cdot x-2 \cdot n-x+4}{n \cdot n-1 \cdot n-2 \cdot n-3}$: now ſuppoſing the Denominator equal to S, multiply the Loſs he will ſuffer, if he has that Chance, by the Probability of having it, and the product $\frac{x \cdot x-1 \cdot x-2 \cdot 4}{S}$ will be his abſolute Loſs in that circumſtance.

Thirdly, If it has paſſed twice, his abſolute Loſs will, by the ſame way of reaſoning, be found to be $\frac{x \cdot x-1 \cdot n-x+2 \cdot 6}{2S}$.

Fourthly, If it has paſſed once, his abſolute Loſs will be found to be $\frac{x \times n-x \cdot n-x-2 \cdot 3}{S}$:

Fifthly, If it has not yet paſſed, his abſolute Loſs will be $\frac{n-x \,.\, n-x-2 \,.\, 2n-2x-5}{2S}$

Now the Sum of all theſe Loſſes of the Ponte will be $\frac{n^3-\frac{9}{2}nn+5n-3x-\frac{3}{2}xx+3x^3}{S}$, and this is the Loſs he ſuffers by venturing a new Stake after any number of Cards x are paſſed.

But the number of Couples which at any time are laid down, is always one half of the number of Cards that are paſſed; wherefore calling t the number of thoſe Couples, the Loſs of the Ponte may be expreſſed thus $\frac{n^3-\frac{9}{2}nn+5n-6t-6tt+24t^3}{S}$

Let now p be the number of Stakes which the Ponte adventures; let alſo the Loſs of the Ponte be divided into two parts, *viz.* $\frac{n^3-\frac{9}{2}nn+5n}{S}$, and $\frac{-6t-6tt+24t}{S}$.

And ſince he adventures a Stake p times; it follows that the firſt part of his loſs will be $\frac{pn^3-\frac{9}{2}pnn+5pn}{S}$.

In order to find the ſecond part, let t be interpreted ſucceſſively by 0, 1, 2, 3, &c. to the laſt term $p-1$; then in the room of $6t$ we ſhall have a Sum of Numbers in Arithmetic Progreſſion to be multiplied by 6; in the room of $6tt$ we ſhall have a Sum of Squares, whoſe Roots are in Arithmetic Progreſſion, to be multiplied by 6; and in the room of $24t^3$ we ſhall have a Sum of Cubes, whoſe Roots are in Arithmetic Progreſſion, to be multiplied by 24.

Theſe ſeveral Sums being collected according to the Rule given in the ſecond Remark on the xth Problem, will be found to be $\frac{6p^4-14p^3+6pp+2p}{S}$ and therefore the whole Loſs of the Ponte will be $\frac{pn^3-\frac{9}{2}pnn+5pn+6p^4-14p^3+6pp+2p}{S}$.

Now this being the Loſs which the Ponte ſuſtains by adventuring the Sum p, each Stake being ſuppoſed equal to Unity, it follows that the Loſs *per Cent.* of the Ponte, is the quantity abovewritten multiplied by 100, and divided by p, which conſidering that S has been ſuppoſed equal to $n \times \overline{n-1} \times \overline{n-2} \times n-3$, will make it to be $\frac{2n-5}{2 \,.\, n-1 \,.\, n-3} \times 100 + \frac{p-1 \times 6pp-8p-2}{n \,.\, n-1 \times n-2 \,.\, n-3} \times 100$;

let now n be interpreted by 52, and p by 23; and the Loſs *per Cent.* of the Ponte, or Gain *per Cent.* of the Banker, will be found to be 2.99251; that is $2^{L.}$ — $19^{\text{ſh.}}$ — $10^{d.}$ *per Cent.*

By the ſame Method of proceſs, it will be found that the Gain *per Cent.* of the Banker at *Baſſette* will be $\frac{3n-9}{n \cdot n-1 \cdot n-2} \times 100 + \frac{4p \cdot p-1 \cdot p-2}{n \cdot n-1 \cdot n-2 \cdot n-3} \times 100$. Let n be interpreted by 51, and p by 23; and the foregoing expreſſion will become 0.790582 or $15^{\text{ſh}}$ — $9\frac{1}{2}^{d.}$. The conſideration of the firſt Stake which is adventured before the Pack is turned being here omitted, as being out of the general Rule; but if that caſe be taken in, the Gain of the Banker will be diminiſhed, and be only 0.76245, that is $15^{\text{ſh.}}$ — $3^{d.}$ very near; and this is to be eſtimated as the Gain *per Cent.* of the Banker, when he takes but half Face.

Now whether the Ponte takes one Card at a time, or ſeveral Cards, the Gain *per Cent.* of the Banker continues the ſame: whether the Ponte keeps conſtantly to the ſame Stake, or ſome time doubles or triples it, the Gain *per Cent.* is ſtill the ſame: whether there be one ſingle Ponte or ſeveral, his Gain *per Cent.* is not thereby altered. Wherefore the Gain *per Cent.* of the Banker, upon all the Money that is adventured at *Pharaon*, is $2^{L.}$ — $19^{\text{ſh.}}$ — $10^{d.}$ and at *Baſſette* $15^{\text{ſh.}}$ — $3^{d.}$

PROBLEM XXXIV.

Suppoſing A *and* B *to play together, that the Chances they have reſpectively to win are as* a *to* b, *and that* B *obliges himſelf to ſet to* A *ſo long as* A *wins without interruption: what is the advantage that* A *gets by his hand?*

SOLUTION.

Firſt, If A and B ſtake 1 each, the Gain of A on the firſt Game is $\frac{a-b}{a+b}$.

Secondly, His Gain on the ſecond Game will alſo be $\frac{a-b}{a+b}$, provided he ſhould happen to win the firſt: but the Probability of A's winning the firſt Game is $\frac{a}{a+b}$. Wherefore his Gain on the ſecond Game will be $\frac{a}{a+b} \times \frac{a-b}{a+b}$.

Thirdly, His Gain on the third Game, after winning the two firſt, will be likewiſe $\frac{a-b}{a+b}$: but the Probability of his winning the two firſt Games is $\frac{aa}{\overline{a+b}|^{2}}$; wherefore his Gain on the third Game is $\frac{aa}{\overline{a+b}|^{2}} \times \frac{a-b}{a+b}$.

Fourthly, Wherefore the Gain of the Hand of *A* is an infinite Series; *viz.* $1 + \frac{a}{a+b} + \frac{aa}{\overline{a+b}|^{2}} + \frac{a^{3}}{\overline{a+b}|^{3}} + \frac{a^{4}}{\overline{a+b}|^{4}}$, &c. to be multiplied by $\frac{a-b}{a+b}$: but the Sum of that infinite Series is $\frac{a+b}{b}$; wherefore the Gain of the Hand of *A* is $\frac{a+b}{b} \times \frac{a-b}{a+b} = \frac{a-b}{b}$.

COROLLARY 1.

If *A* has the advantage of the Odds, and *B* ſets his hand out, the Gain of *A* is the difference of the numbers expreſſing the Odds, divided by the leſſer. Thus if *A* has the Odds of 5 to 3, then his Gain will be $\frac{5-3}{3} = \frac{2}{3}$.

COROLLARY 2.

If *B* has the diſadvantage of the Odds, and *A* ſets his hand out, the Loſs of *B* will be the difference of the numbers expreſſing the Odds divided by the greater: thus if *B* has but 3 to 5, his Loſs will be $\frac{2}{5}$.

COROLLARY 3.

If *A* and *B* do mutually engage to ſet to one another, as long as either of them wins without interruption, the Gain of *A* will be found to be $\frac{aa-bb}{ab}$; that is the Sum of the numbers expreſſing the Odds multiplied by their difference, the Product of that Multiplication being divided by the Product of the numbers expreſſing the Odds. Thus if the Odds were as 5 to 3, the Sum of 5 and 3 being 8, and the difference being 2, multiply 8 by 2, and divide the product 16, by the product of the numbers expreſſing the Odds, which is 15, and the Quotient will be $\frac{16}{15}$, or $1\frac{1}{15}$, which therefore will be the Gain of *A*.

COROLLARY 4.

But if he be only to be ſet to, who wins the firſt time, and that he be to be ſet to as long as he wins without interruption; then the

Gain of *A* will be $\frac{a^3-b^3}{ab\times\overline{a+b}}$: thus if a be 5, and b 3, the Gain of *A* will be $\frac{98}{120}=\frac{49}{60}$.

PROBLEM XXXV.

Any number of Letters a, b, c, d, e, f, *&c. all of them different, being taken promiſcuouſly as it happens: to find the Probability that ſome of them ſhall be found in their places according to the rank they obtain in the Alphabet; and that others of them ſhall at the ſame time be diſplaced.*

SOLUTION.

Let the number of all the Letters be $=n$; let the number of thoſe that are to be in their places be $=p$, and the number of thoſe that are to be out of their places $=q$. Suppoſe for brevity's ſake $\frac{1}{n}=r$, $\frac{1}{n\,.\,n-1}=s$, $\frac{1}{n\,.\,n-1\,.\,n-2}=t$, $\frac{1}{n\,.\,n-1\,.\,n-2\,.\,n-3}$ $=v$, &c. then let all the quantities 1, r, s, t, v, &c. be written down with Signs alternately poſitive and negative, beginning at 1, if p be $=0$; at r, if p be $=1$: at s, if p be $=2$, &c. Prefix to theſe Quantities the Coefficients of a Binomial Power, whoſe index is $=q$; this being done, thoſe Quantities taken all together will expreſs the Probability required. Thus the Probability that in 6 Letters taken promiſcuouſly, two of them, *viz.* a and b ſhall be in their places, and three of them, *viz.* c, d, e, out of their places, will be

$$\frac{1}{6\,.\,5}-\frac{3}{6\,.\,5\,.\,4}+\frac{3}{6\,.\,5\,.\,4\,.\,3}-\frac{1}{6\,.\,5\,.\,4\,.\,3\,.\,2}=\frac{11}{720}$$

And the Probability that a ſhall be in its place, and b, c, d, e, out of their places, will be

$$\frac{1}{6}-\frac{4}{6\,.\,5}+\frac{6}{6\,.\,5\,.\,4}-\frac{4}{6\,.\,5\,.\,4\,.\,3}+\frac{1}{6\,.\,5\,.\,4\,.\,3\,.\,2}=\frac{53}{720}$$

The Probability that a ſhall be in its place, and b, c, d, e, f, out of their places, will be

$$\frac{1}{6}-\frac{5}{6\,.\,5}+\frac{10}{6\,.\,5\,.\,4}-\frac{10}{6\,.\,5\,.\,4\,.\,3}+\frac{5}{6\,.\,5\,.\,4\,.\,3\,.\,2}-\frac{1}{6\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1}$$
$$=\frac{44}{720}=\frac{11}{180}.$$

The Probability that *a*, *b*, *c*, *d*, *e*, *f* ſhall all be diſplaced is

$$1 - \frac{6}{6} + \frac{15}{6\cdot5} - \frac{20}{6\cdot5\cdot4} + \frac{15}{6\cdot5\cdot4\cdot3} - \frac{6}{6\cdot5\cdot4\cdot3\cdot2} + \frac{1}{6\cdot5\cdot4\cdot3\cdot2\cdot1} \text{ or } 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} = \frac{265}{720} = \frac{53}{144}.$$

Hence it may be concluded, that the Probability that one of them at leaſt ſhall be in its place, is $1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} = \frac{91}{144}$, and that the Odds that one of them at leaſt ſhall be ſo found, are as 91 to 53.

It muſt be obſerved, that the foregoing Expreſſion may ſerve for any number of Letters, by continuing it to ſo many Terms as there are Letters: thus if the number of Letters had been ſeven, the Probability required would have been $\frac{177}{280}$.

DEMONSTRATION.

The number of Chances for the Letter *a* to be in the firſt place, contains the number of Chances by which *a* being in the firſt place, *b* may be in the ſecond, or out of it: This is an Axiom of common Senſe of the ſame degree of evidence, as that the whole is equal to all its parts.

From this it follows, that if from the number of Chances that there are for *a* to be in the firſt place, there be ſubtracted the number of Chances that there are for *a* to be in the firſt place, and *b* at the ſame time in the ſecond, there will remain the number of Chances by which *a* being in the firſt place, *b* may be excluded the ſecond.

For the ſame reaſon it follows, that if from the number of Chances for *a* and *b* to be reſpectively in the firſt and ſecond places, there be ſubtracted the number of Chances by which *a*, *b*, and *c* may be reſpectively in the firſt, ſecond, and third places; there will remain the number of Chances by which *a* being in the firſt, and *b* in the ſecond, *c* may be excluded the third place: and ſo of the reſt.

Let $+a$ denote the Probability that *a* ſhall be in the firſt place, and let $-a$ denote the Probability of its being out of it. Likewiſe let the Probabilities that *b* ſhall be in the ſecond place, or out of it, be reſpectively expreſt by $+b$ and $-b$.

Let the Probability that a being in the firſt place, b ſhall be in the ſecond, be expreſſed by $a + b$: Likewiſe let the Probability that a being in the firſt place, b ſhall be excluded the ſecond, be expreſſed by $a - b$.

Univerſally. Let the Probability there is that as many as are to be in their proper places, ſhall be ſo, and that as many others as are at the ſame time to be out of their proper places ſhall be ſo found, be denoted by the particular Probabilities of their being in their proper places, or out of them, written all together: So that, for inſtance $a + b + c - d - e$, may denote the Probability that a, b, and c ſhall be in their proper places, and that at the ſame time both d and e ſhall be excluded their proper places.

Now to be able to derive proper Concluſions by virtue of this Notation, it is to be obſerved, that of the Quantities which are here conſidered, thoſe from which the Subtraction is to be made are indifferently compoſed of any number of Terms connected by $+$ and $-$; and that the Quantities which are to be ſubtracted do exceed by one Term thoſe from which the Subtraction is to be made; the reſt of the Terms being alike, and their Signs alike; then nothing more is requiſite to have the remainder, than to preſerve the Quantities that are alike, with their proper Signs, and to change the Sign of the Quantity exceeding.

It having been demonſtrated in what we have ſaid of Permutations and Combinations, that $a = \frac{1}{n}$; $a + b = \frac{1}{n \, . \, n-1}$; $a + b + c = \frac{1}{n \, . \, n-1 \, . \, n-2}$, &c. let $\frac{1}{n}$, $\frac{1}{n \, . \, n-1}$, &c. be reſpectively called r, s, t, v, &c. this being ſuppoſed, we may come to the following Concluſions.

$$\begin{array}{rll}
 & b \qquad = r & \\
 & b + a = s & \\ \hline
\text{then } 1^\circ. & b - a = r - s & \\ \hline
 & c + b \qquad = s & \text{for the ſame reaſon that } a + b = s. \\
 & c + b + a = t & \\ \hline
2^\circ. & c + b - a = s - t & \\ \hline
 & c - a \qquad = r - s & \text{by the firſt Concluſion.} \\
 & c - a + b = \; + s - t & \text{by the ſecond.} \\ \hline
3^\circ. & c - a - b = r - 2s + t & \\ \hline
 & d + c + b = t & \\
 & d + c + b + a = v & \\ \hline
4^\circ. & d + c + b - a = t - v & \\ \hline
\end{array}$$

$d + c - a = s - t$ by the ſecond Concluſion.
$d + c - a + b = t - v$ by the fourth.
5°. $d + c - a - b = s - 2t + v$
$d - b - a = r - 2s + t$ by the third Concluſion.
$d - b - a + c = s - 2t + v$ by the fifth.
6°. $d - b - a - c = r - 3s + 3t - v$.

By the ſame proceſs, if no letter be particularly aſſigned to be in its place the Probability that ſuch of them as are aſſigned may be out of their places, will likewiſe be found thus.

$- a = 1 - r$, for $+ a$ and $- a$ together make Unity.
$- a + b = r - s$ by the firſt Concluſion.
7°. $- a - b = 1 - 2r + s$
$- a - b = 1 - 2r + s$ by the ſeventh Concluſion.
$- a - b + c = r - 2s + t$ by the third.
8°. $- a - b - c = 1 - 3r + 3s - t$.

Now examining carefully all the foregoing Concluſions, it will be perceived, that when a Queſtion runs barely upon the diſplacing any given number of Letters, without requiring that any other ſhould be in its place, but leaving it wholly indifferent; then the Vulgar Algebraic Quantities which lie at the right-hand of the Equations, begin conſtantly with Unity: it will alſo be perceived, that when one ſingle Letter is aſſigned to be in its place, then thoſe Quantities begin with r, and that when two Letters are aſſigned to be in their places, they begin with s, and ſo on: moreover 'tis obvious, that theſe Quantities change their Signs alternately, and that the numerical Coefficients, which are prefixed to them are thoſe of a Binomial Power, whoſe Index is equal to the number of Letters which are to be diſplaced.

PROBLEM XXXVI.

Any given number of Letters a, b, c, d, e, f, *&c. being each repeated a certain number of times, and taken promiſcuouſly as it happens: To find the Probability that of ſome of thoſe ſorts, ſome one Letter of each may be found in its place, and at the ſame time, that of ſome other ſorts, no one Letter be found in its place.*

SOLUTION.

Suppoſe n be the number of all the Letters, l the number of times that each Letter is repeated, and conſequently $\frac{n}{l}$ the whole number of Sorts: ſuppoſe alſo that p be the number of Sorts of which ſome one Letter is to be found in its place, and q the number of Sorts of which no one Letter is to be found in its place. Let now the preſcriptions given in the preceding Problem be followed in all reſpects, ſaving that r muſt here be made $= \frac{l}{n}$, $s = \frac{ll}{n \,.\, n-1}$, $t = \frac{l^3}{n \,.\, n-1 \,.\, n-2}$, &c. and the Solution of any particular caſe of the Problem will be obtained.

Thus if it were required to find the Probability that no Letter of any ſort ſhall be in its place, the Probability thereof would be expreſſed by the Series

$$1 - qr + \frac{q \,.\, q-1}{1 \,.\, 2} s - \frac{q \,.\, q-1 \,.\, q-2}{1 \,.\, 2 \,.\, 3} + \frac{n \,.\, q-1 \,.\, q-2 \,.\, q-3}{1 \,.\, 2 \,.\, 3 \,.\, 4} v, \text{ \&c.}$$

of which the number of Terms is equal to $q + 1$.

But in this particular caſe q would be equal to $\frac{n}{l}$, and therefore, the foregoing Series might be changed into this, *viz.*

$$\frac{1}{2} \times \frac{n-l}{n-1} - \frac{1}{6} \times \frac{n-l \,.\, n-2l}{n-1 \,.\, n-2} + \frac{1}{24} \times \frac{n-l \,.\, n-2l \,.\, n-3l}{n-1 \,.\, n-2 \,.\, n-3}, \text{ \&c.}$$

of which the number of Terms is equal to $\frac{n-l}{l}$.

COROLLARY 1.

From hence it follows, that the Probability of one or more Letters, indeterminately taken, being in their places, will be expreſſed as follows:

$$1 - \frac{1}{2} \times \frac{n-l}{n-1} + \frac{1}{6} \times \frac{n-l \,.\, n-2l}{n-1 \,.\, n-2} - \frac{1}{24} \times \frac{n-l \,.\, n-2l \,.\, n-3l}{n-1 \,.\, n-2 \,.\, n-3}, \text{ \&c.}$$

COROLLARY 2.

The Probability of two or more Letters indeterminately taken, being in their places, will be

$\frac{1}{2} \times \frac{n-l}{n-1} - \frac{2}{1 \,.\, 3} \times \frac{n-2l}{n-2} A + \frac{3}{2 \,.\, 4} \times \frac{n-3l}{n-3} B - \frac{4}{3 \,.\, 5} \times \frac{n-4l}{n-4} C + \frac{5}{4 \,.\, 6} \times \frac{n-5l}{n-5} D$, &c. wherein it is neceſſary to obſerve, that the Capitals A, B, C, D, &c. denote the preceding Terms.

Altho' the formation of this laſt Series flows naturally from what we have already eſtabliſhed, yet that nothing may be wanting to clear up this matter, it is to be obſerved, that if one Species is to have ſome one of its Letters in its proper place, and the reſt of the Species to be excluded, then the Series whereby the Problem is determined being to begin at r, according to the Precepts given in the preceding Problem, becomes

$$r - qs + \frac{q\,.\,q-1}{1\,.\,2}t - \frac{q\,.\,q-1\,.\,q-2}{1\,.\,2\,.\,3}v, \text{ \&c.}$$

but then the number of Species being $\frac{n}{l}$, and all but one being to be excluded, it follows that q in this caſe is $= \frac{n}{l} - 1 = \frac{n-l}{l}$ wherefore the preceding Series would become, after the proper Subſtitutions,

$$\frac{l}{n} - \frac{n-l\,.\,l}{n\,.\,n-1} + \frac{1}{2} \times \frac{n-l\,.\,n-2l\,.\,l}{n\,.\,n-1\,.\,n-2} - \frac{1}{6} \times \frac{n-l\,.\,n-2l\,.\,n-3l\,.\,l}{n\,.\,n-1\,.\,n-2\,.\,n-3}, \text{ \&c.}$$

And this is the Probability that ſome one of the Letters of the Species particularly given, may obtain its place, and the reſt of the Species be excluded; but the number of Species being $\frac{n}{l}$, it follows that this Series ought to be multiplied by $\frac{n}{l}$, which will make it

$$1 - \frac{n-l}{n-1} + \frac{1}{2} \times \frac{n-l\,.\,n-2l}{n-1\,.\,n-2} - \frac{1}{6} \times \frac{n-l\,.\,n-2l\,.\,n-3l}{n-1\,.\,n-2\,.\,n-3}, \text{ \&c.}$$

And this is the Probability that ſome one Species indeterminately taken, and no more than one, may have ſome one of its Letters in its proper place.

Now if from the Probability of one or more being in their places, be ſubtracted the Probability of one and no more being in its place, there will remain the Probability of two or more indeterminately taken being in their places, which conſequently will be the difference between the following Series, *viz.*

$$1 - \frac{1}{2} \times \frac{n-l}{n-1} + \frac{1}{6} \times \frac{n-l\,.\,n-2l}{n-1\,.\,n-2} - \frac{1}{24} \times \frac{n-l\,.\,n-2l\,.\,n-3l}{n-1\,.\,n-2\,.\,n-3}, \text{ \&c.}$$

$$\text{and } 1 - \frac{n-l}{n-1} + \frac{1}{2} \times \frac{n-l\,.\,n-2l}{n-1\,.\,n-2} - \frac{1}{6} \times \frac{n-l\,.\,n-2l\,.\,n-3l}{n-1\,.\,n-2\,.\,n-3}, \text{ \&c.}$$

which difference therefore will be

$\frac{1}{2}$

$$\frac{1}{2}\times\frac{n-l}{n-1}-\frac{1}{3}\times\frac{n-l\,.\,n-2l}{n-1\,.\,n-2}+\frac{1}{8}\times\frac{n-l\,.\,n-2l\,.\,n-3l}{n-1\,.\,n-2\,.\,n-3}, \text{ \&c.}$$

or $\frac{1}{2}\times\frac{n-l}{n-1}-\frac{2}{1\,.\,3}\times\frac{n-2l}{n-2}A+\frac{3}{2\,.\,4}\times\frac{n-3l}{n-3}B-\frac{4}{3\,.\,5}\times\frac{n-4l}{n-4}C$, &c.

as we had expreſſed it before: and from the ſame way of reaſoning, the other following Corollaries may be deduced.

COROLLARY 3.

The Probability that three or more Letters indeterminately taken may be in their places, will be expreſſed by the Series

$$\frac{1}{6}\times\frac{n-l\,.\,n-2l}{n-1\,.\,n-2}-\frac{3}{1\,.\,4}\times\frac{n-3l}{n-3}A+\frac{4}{2\,.\,5}\times\frac{n-4l}{n-4}B$$
$$-\frac{5}{3\,.\,6}\times\frac{n-5l}{n-5}C+\frac{6}{4\,.\,7}\times\frac{n-6l}{n-6}D, \text{ \&c.}$$

COROLLARY 4.

The Probability that four or more Letters indeterminately taken may be in their places will be thus expreſſed

$$\frac{1}{24}\times\frac{n-l\,.\,n-2l\,.\,n-3l}{n-1\,.\,n-2\,.\,n-3}-\frac{4}{1\,.\,5}\times\frac{n-4l}{n-4}A+\frac{5}{2\,.\,6}\times\frac{n-5l}{n-5}B$$
$$-\frac{6}{3\,.\,7}\times\frac{n-6l}{n-6}C, \text{ \&c.}$$

The Law of the continuation of theſe Series being manifeſt, it will always be eaſy to aſſign one that ſhall fit any caſe propoſed.

From what we have ſaid it follows, that in a common Pack of 52 Cards, the Probability that one of the four Aces may be in the firſt place, or one of the four Deuces in the ſecond, or one of the four Trays in the third; or that ſome of any other ſort may be in its place (making 13 different places in all) will be expreſſed by the Series exhibited in the firſt Corollary.

It follows likewiſe, that if there be two Packs of Cards, and that the order of the Cards in one of the Packs be the Rule whereby to eſtimate the rank which the Cards of the ſame Suit and Name are to obtain in the other; the Probability that one Card or more in one of the Packs may be found in the ſame poſition as the like Card in the other Pack, will be expreſſed by the Series belonging to the firſt Corollary, making $n=52$, and $l=1$. Which Series will in this caſe be $1-\frac{1}{2}+\frac{1}{6}-\frac{1}{24}+\frac{1}{120}-\frac{1}{720}$, &c. whereof 52 Terms are to be taken.

If the Terms of the foregoing Series are joined by Couples, the Series will become

$$\frac{1}{2} + \frac{1}{2\cdot 4} + \frac{1}{2\cdot 3\cdot 4\cdot 6} + \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 8} + \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 10}$$

&c. of which 26 Terms ought to be taken.

But by reaſon of the great Convergency of the ſaid Series, a few of its Terms will give a ſufficient approximation, in all caſes; as appears by the following Operation

$$\begin{array}{r}
\frac{1}{2} = 0.500000 \\
\frac{1}{2\cdot 4} = 0.125000 \\
\frac{1}{2\cdot 3\cdot 4\cdot 6} = 0.006944 + \\
\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 8} = 0.000174 + \\
\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 10} = 0.000002 + \\
\hline
\text{Sum} = 0.632120 +
\end{array}$$

Wherefore the Probability that one or more like Cards in two different Packs may obtain the ſame poſition, is very nearly 0.632, and the Odds that this will happen once at leaſt as 632 to 368, or 12 to 7 very near.

But the Odds that two or more like Cards in two different Packs will not obtain the ſame poſition are very nearly as 736 to 264, or 14 to 5.

REMARK.

It is known that $1 + y + \frac{1}{2}yy + \frac{1}{6}y^3 + \frac{1}{24}y^4$ &c. is the number whoſe hyperbolic Logarithm is y, and therefore $1 - y + \frac{1}{2}yy - \frac{1}{6}y^3 + \frac{1}{24}y^4$ &c. is the Number whoſe hyperbolic Logarithm is $-y$. Let N be $= y - \frac{1}{2}yy + \frac{1}{6}y^3 - \frac{1}{24}y^4$ &c. then $1 - N$ is the Number whoſe hyperbolic Logarithm is $-y$. Let now y be $= 1$, therefore $1 - N$ is the number whoſe hyperbolic Logarithm is -1; but the number whoſe hyperbolic Logarithm is -1, is the reciprocal of that whoſe hyperbolic Logarithm is 1, or whoſe *Briggian* Logarithm is 0.4342944. Therefore 9.5657056 is the *Briggian* Logarithm anſwering to the hyperbolic Logarithm -1, but the number anſwering to it is 0.36788. Therefore $1 - N = 0.36788$; and $N = 1 - 0.36788 = 0.63212$; and therefore the Series $y - \frac{1}{2}yy + \frac{1}{6}y^3 - \frac{1}{24}y^4$ &c. *in infinitum*, when $y = 1$, that is $1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24}$ &c. $= 0.63212$.

COROLLARY 5.

If *A* and *B* each holding a Pack of Cards, pull them out at the ſame time one after another, on condition that every time two like Cards are pulled out, *A* ſhall give *B* a Guinea; and it were required to find what conſideration *B* ought to give *A* to play on thoſe Terms: the Anſwer will be one Guinea, let the number of Cards be what it will.

Altho' this be a Corollary from the preceding Solutions, yet it may more eaſily be made out thus; one of the Packs being the Rule whereby to eſtimate the order of the Cards in the ſecond, the Probability that the two firſt Cards are alike is $\frac{1}{52}$, the Probability that the two ſecond are alike is alſo $\frac{1}{52}$, and therefore there being 52 ſuch alike combinations, it follows that the Value of the whole is $\frac{52}{52} = 1$.

COROLLARY 6.

If the number of Packs be given, the Probability that any given number of Circumſtances may happen in any number of Packs, will eaſily be found by our Method: thus if the number of Packs be *k*, the Probability that one Card or more of the ſame Suit and Name in every one of the Packs may be in the ſame poſition, will be expreſſed as follows,

$$\frac{1}{n^{k-2}} - \frac{1}{\overline{2 . n . n-1}\Big|^{k-2}} + \frac{1}{\overline{6 . n . n-1 ; n-1}\Big|^{k-2}} - \frac{1}{\overline{24 . n . n-1 . n-2 . n-3}\Big|^{k-2}}, \ \&c.$$

PROBLEM XXXVII.

If A *and* B *play together each with a certain number of Bowls* = n: *what are the reſpective Probabilities of winning, ſuppoſing that each of them want a certain number of Games of being up?*

SOLU-

SOLUTION.

Firſt, The Probability that ſome Bowl of *B* may be nearer the Jack than any Bowl of *A* is $\frac{1}{2}$; *A* and *B* in this Problem being ſuppoſed equal Players.

Secondly, Suppoſing one of his Bowls nearer the Jack than any Bowl of *A*, the number of his remaining Bowls is $n - 1$, and the number of all the Bowls remaining between them is $2n - 1$: wherefore the Probability that ſome other of his Bowls may be nearer the Jack than any Bowl of *A* will be $\frac{n-1}{2n-1}$, from whence it follows, that the Probability of his winning two Bowls or more is $\frac{1}{2} \times \frac{n-1}{2n-1}$.

Thirdly, Suppoſing two of his Bowls nearer the Jack than any Bowl of *A*, the Probability that ſome other of his Bowls may be nearer the Jack than any Bowl of *A*, will be $\frac{n-2}{2n-2}$; wherefore the Probability of his winning three Bowls or more is $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n-2}{2n-2}$: the continuation of which proceſs is manifeſt.

Fourthly, The Probability that one ſingle Bowl of *B* ſhall be nearer the Jack than any Bowl of *A* is $\frac{1}{2} - \frac{1}{2} \times \frac{n-1}{2n-1}$ or $\frac{1}{2} \times \frac{n}{2n-1}$; for if from the Probability that one or more of his Bowls may be nearer the Jack than any Bowl of *A*, there be ſubtracted the Probability that two or more may be nearer, there remains the probability of one ſingle Bowl of *B* being nearer: in this caſe *B* is ſaid to win one Bowl at an end.

Fifthly, The Probability that two Bowls of *B*, and not more, may be nearer the Jack than any Bowl of *A*, will be found to be $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n}{2n-2}$, in which caſe, *B* is ſaid to win two Bowls at an end.

Sixthly, The Probability that *B* may win three Bowls at an end will be found to be $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n-2}{2n-2} \times \frac{n}{2n-3}$; the proceſs whereof is manifeſt.

It may be obſerved, that the foregoing Expreſſions might be reduced to fewer Terms; but leaving them unreduced, the Law of the Proceſs is thereby made more conſpicuous.

It is carefully to be obſerved, when we mention henceforth the probability of winning two Bowls, that the Senſe of it ought to be extended to two Bowls or more; and that when we mention the winning two Bowls at an end, it ought to be taken in the common acceptation of two Bowls only: the like being to be obſerved in other caſes.

This preparation being made; ſuppoſe that *A* wants one Game of being up and *B* two; and it be required in that circumſtance to determine their probabilities of winning.

Let the whole Stake between them be ſuppoſed $= 1$. Then either *A* may win a Bowl, or *B* win one Bowl at an end, or *B* may win two Bowls.

In the firſt caſe be loſes his Expectation.

In the ſecond caſe *B* becomes intitled to $\frac{1}{2}$ of the Stake, but the probability of this caſe is $\frac{1}{2} \times \frac{n}{2n-1}$. Wherefore his Expectation ariſing from that part of the Stake he will be intitled to, if this Caſe ſhould happen, and from the probability of its happening, will be $\frac{1}{4} \times \frac{n}{2n-1}$.

In the third caſe, *B* wins the whole Stake 1, but the probability of this Caſe, is $\frac{1}{2} \times \frac{n-1}{2n-1}$.

From this it follows, that the whole Expectation of *B* is equal to $\frac{1}{4} \times \frac{n}{2n-1} + \frac{1}{2} \times \frac{n-1}{2n-1}$, or $\frac{3n-2}{8n-4}$. Which being ſubtracted from Unity, the remainder will be the Expectation of *A*, *viz.* $\frac{5n-2}{8n-4}$. It may therefore be concluded that the Probabilities which *A* and *B* have of winning are reſpectively as $5n-2$ to $3n-2$.

'Tis remarkable, that the fewer the Bowls are the greater is the proportion of the Odds; for if *A* and *B* play with ſingle Bowls, the proportion will be as 3 to 1; if they play with two Bowls each, the proportion will be as 2 to 1; if they play with three Bowls each, the proportion will be as 13 to 6; yet let the number of Bowls be never ſo great, that proportion will not deſcend ſo low as 5 to 3.

Let us now ſuppoſe that *A* wants one Game of being up, and *B* three; then either *A* may win a Bowl, or *B* one Bowl at an end, or two Bowls at an end, or three Bowls.

In the firſt Caſe, *B* loſes his Expectation.

If the ſecond Caſe happen, then *B* will be in the circumſtance of wanting but two to *A*'s one; in which caſe his Expectation will be $\frac{3n-2}{8n-4}$, as it has been before determined: but the probability that this Caſe may happen is $\frac{1}{2} \times \frac{n}{2n-1}$; wherefore the Expectation of *B* ariſing from the proſpect of this Caſe will be equal to $\frac{1}{2} \times \frac{n}{2n-1} \times \frac{3n-2}{8n-4}$.

If the third Caſe happen, then *B* will be intitled to one half of the Stake: but the Probability of its happening is $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n}{2n-2}$; wherefore the Expectation of *B* ariſing from the proſpect of this caſe is $\frac{1}{4} \times \frac{n-1}{2n-1} \times \frac{n}{2n-2}$ or $\frac{1}{8} \times \frac{n}{2n-1}$.

If the fourth Caſe happen, then *B* wins the whole Stake 1: but the Probability of its happening is $\frac{1}{2} \times \frac{n-1}{2n-1} \times \frac{n}{2n-2}$ or $\frac{1}{4} \times \frac{n-2}{2n-1}$

From this it follows, that the whole Expectation of *B* will be $\frac{9nn-13n+4}{8 \times \overline{2n-1}|^{2}}$; which being ſubtracted from Unity, the remainder will be the Expectation of *A*, *viz.* $\frac{23nn-19n+4}{8 \times \overline{2n-1}|^{2}}$. It may therefore be concluded that the Probabilities which *A* and *B* have of winning will be as $23nn-19n+4$ to $9nn-13n+4$.

N. B. If *A* and *B* play only with one Bowl each, the Expectation of *B* as deduced from the foregoing Theorem would be found $=0$, which we know from other principles ought to be $=\frac{1}{8}$. The reaſon of which is, that the caſe of winning two Bowls at an end, and the caſe of winning three Bowls enter the general concluſion, which caſes do not belong to the Suppoſition of playing with ſingle Bowls; wherefore excluding thoſe two Caſes, the Expectation of *B* will be found to be $\frac{1}{2} \times \frac{n}{2n-1} \times \frac{3n-2}{8n-4} = \frac{1}{8}$, which will appear if n be made $=1$. But the Expectation of *B* in the caſe of two Bowls would be rightly determined from the general Solution: the reaſon of which is, that the Probability of winning three Bowls being univerſally $\frac{1}{4} \times \frac{n-2}{2n-1}$, that Expreſſion becomes $=0$, when n is interpreted by 2; which makes it that the general Expreſſion is applicable to this Caſe.

After what has been ſaid, it will be eaſy to extend this way of reaſoning to any circumſtance of Games wanting between *A* and *B*; by making the Solution of each ſimpler Caſe ſubſervient to the Solution of that which is next more compounded.

Having given formerly the Solution of this Problem, propoſed to me by the Honourable *Francis Robartes* Eſq;, in the *Philoſophical Tranſactions* Number 329; I there ſaid, by way of Corollary, that if the proportion of Skill in the Gameſters were given, the Problem might alſo be ſolved: ſince which time M. *de Monmort*, in the ſecond Edition of a Book by him publiſhed upon the Subject of Chance, has ſolved this Problem as it is extended to the conſideration of the Skill, and to carry his Solution to a great number of Caſes, giving alſo a Method whereby it might be carried farther: But altho' his Solution is good, as he has made a right uſe of the Doctrine of Combinations, yet I think mine has a greater degree of Simplicity, it being deduced from the original Principle whereby I have demonſtrated the Doctrine of Permutations and Combinations: wherefore to make it as familiar as poſſible, and to ſhew its vaſt extent, I ſhall now apply it to the general Solution of this Problem, by taking in the conſideration of the Skill of the Gameſters.

But before I proceed, it is neceſſary to define what I call *Skill: viz.* that it is the proportion of Chances which the Gameſters may be ſuppoſed to have for winning a ſingle Game with one Bowl each.

PROBLEM XXXVIII.

If A *and* B, *whoſe proportion of ſkill is as* a *to* b, *play together each with a certain number of Bowls: what are their reſpective Probabilities of winning, ſuppoſing each of them to want a certain number of Games of being up?*

SOLUTION.

Firſt, The Chance of *B* for winning one ſingle Bowl being b, and the number of his Bowls being n, it follows that the Sum of all his Chances is nb; and for the ſame reaſon, the Sum of all the Chances of *A* is na: wherefore the Sum of all the Chances for winning one Bowl or more is $na+nb$; which for brevity's ſake we may call s. From whence it follows, that the Probability which *B* has of winning one Bowl is $\frac{nb}{s}$.

Secondly, Suppoſing one of his Bowls nearer the Jack than any of the Bowls of *A*, the number of his remaining Chances is $\overline{n-1}\times b$; and the number of Chances remaining between them is $s-b$: wherefore the Probability that ſome other of his Bowls may be nearer the Jack than any Bowl of *A* will be $\frac{\overline{n-1}\times b}{s-b}$; from whence it follows that the Probability of his winning two Bowls or more is $\frac{nb}{s}\times\frac{\overline{n-1}\times b}{s-b}$.

Thirdly, Suppoſing two of his Bowls nearer the Jack than any of the Bowls of *A*, the number of his remaining Chances is $\overline{n-2}\times b$; and the number of Chances remaining between them is $s-2b$; wherefore the Probability that ſome other of his Bowls may be nearer the Jack than any Bowl of *A* will be $\frac{\overline{n-2}\times b}{s-2b}$. From whence it follows that the Probability of his winning three Bowls or more is $\frac{nb}{s}\times\frac{\overline{n-1}\times b}{s-b}\times\frac{\overline{n-2}\times b}{s-2b}$; the continuation of which proceſs is manifeſt.

Fourthly, If from the Probability which *B* has of winning one Bowl or more, there be ſubtracted the Probability which he has of winning two or more, there will remain the Probability of his winning one Bowl at an end: which therefore will be found to be

$$\frac{nb}{s}-\frac{nb}{s}\times\frac{\overline{n-1}\times b}{s-b},\ \text{or}\ \frac{nb}{s}\times\frac{s-nb}{s-b},\ \text{or}\ \frac{nb}{s}\times\frac{an}{s-b}.$$

Fifthly, For the ſame reaſons as above, the Probability which *B* has of winning two Bowls at an end, will be $\frac{nb}{s}\times\frac{\overline{n-1}\times b}{s-b}\times\frac{an}{s-2b}$.

Sixthly, And for the ſame reaſon likewiſe, the Probability which *B* has of winning three Bowls at an end will be found to be $\frac{nb}{s}\times\frac{\overline{n-1}\times b}{s-b}\times\frac{\overline{n-2}\times b}{s-2b}\times\frac{an}{s-3b}$; The continuation of which proceſs is manifeſt.

N. B. The ſame Expectations which denote the Probability of any circumſtance of *B* will denote likewiſe the Probability of the like circumſtance of *A*, only changing *b* into *a*, and *a* into *b*.

Theſe things being premiſed, ſuppoſe *firſt*, that each wants one Game of being up; 'tis plain, that the Expectations of *A* and *B* are reſpectively $\frac{an}{s}$ and $\frac{bn}{s}$. Let this Expectation of *B* be called P.

Secondly, Suppoſe *A* wants one Game of being up, and *B* two, and let the Expectation of *B* be required: then either *A* may win a Bowl, or *B* win one Bowl at an end, or *B* win two Bowls.

If the firſt Caſe happens, *B* loſes his Expectation.

If the ſecond happens, he gets the Expectation P; but the Probability of this Caſe is $\frac{nb}{s}\times\frac{an}{s-b}$: wherefore the Expectation of *B* ariſing from the poſſibility that it may ſo happen is $\frac{nb}{s}\times\frac{an}{s-b}\times$ P.

If the third Caſe happens, he gets the whole Stake 1; but the Probability of this Caſe is $\frac{nb}{s} \times \frac{nb-b}{s-b}$; wherefore the Expectation of *B* ariſing from the Probability of this Caſe is $\frac{nb}{s} \times \frac{nb-b}{s-b} \times 1$.

From which it follows, that the whole Expectation of *B* will be $\frac{nb}{s} \times \frac{an}{s-b} \times \mathrm{P} + \frac{nb}{s} \times \frac{nb-b}{s-b}$. Let this Expectation be called Q.

Thirdly, Suppoſe *A* to want one Game of being up, and *B* three: then either *B* may win one Bowl at an end, in which Caſe he gets the Expectation Q; or two Bowls at an end, in which Caſe he gets the Expectation P; or three Bowls, in which Caſe he gets the whole Stake 1. Wherefore the Expectation of *B* will be foned to be $\frac{nb}{s} \times \frac{an}{s-b} \times \mathrm{Q} + \frac{nb}{s} \times \frac{\overline{n-1} \times b}{s-b} \times \frac{an}{s-2b} \times \mathrm{P} + \frac{nb}{s} \times \frac{\overline{n-1} \times b}{s-b} \times \frac{\overline{n-2} \times b}{s-2b}$.

An infinite number of theſe Theorems may be formed in the ſame manner, which may be continued by inſpection, having well obſerved how each of them is deduced from the preceding.

If the number of Bowls were unequal, ſo that *A* had *m* Bowls, and *B*, *n* Bowls; then ſuppoſing $ma + nb = s$, other Theorems might be found to anſwer that inequality: and if that inequality ſhould not be conſtant, but vary at pleaſure; other Theorems might alſo be found to anſwer that Variation of inequality, by following the ſame way of arguing. And if three or more Gameſters were to play together under any circumſtance of Games wanting, and of any given proportion of Skill, their Probabilities of winning might be determined in the ſame manner.

PROBLEM XXXIX.

To find the Expectation of A, *when with a Die of any given number of Faces, he undertakes to fling any number of them in any given number of Caſts.*

SOLUTION.

Let $p + 1$ be the number of all the Faces in the Die, n the number of Caſts, f the number of Faces which he undertakes to fling.

The number of Chances for the Ace to come up once or more in any number of Caſts n, is $\overline{p+1}|^n - p^n$: as has been proved in the Introduction.

Let the *Deux*, by thought, be expunged out of the Die, and thereby the number of its Faces reduced to p, then the number of Chances for the *Ace* to come up will at the ſame time be reduced to $p^n - \overline{p-1}|^n$. Let now the *Deux* be reſtored, and the number of Chances for the *Ace* to come up without the *Deux*, will be the ſame as if the *Deux* were expunged: But if from the number of Chances for the *Ace* to come up with or without the *Deux*, *viz.* from $\overline{p+1}|^n - p^n$ be ſubtracted the number of Chances for the *Ace* to come up without the *Deux*, *viz.* $p^n - \overline{p-1}|^n$, there will remain the number of Chances for the *Ace* and the *Deux* to come up once or more in the given number of Caſts, which number of Chances conſequently will be $\overline{p+1}|^n - 2p^n + \overline{p-1}|^n$.

By the ſame way of arguing it will be proved, that the number of Chances, for the *Ace* and *Deux* to come up without the *Tray*, will be $p^n - 2 \times \overline{p-1}|^n + \overline{p-2}|^n$, and conſequently that the number of Chances for the *Ace*, the *Deux*, and *Tray* to come up once or more, will be the difference between $\overline{p+1}|^n - 2p^n + \overline{p-1}|^n$, and $p^n - 2 \times \overline{p-1}|^n + \overline{p-2}|^n$, which therefore will be $\overline{p+1}|^n - 3 \times p^n + 3 \times \overline{p-1}|^n - \overline{p-2}|^n$.

Again, it may be proved that the number of Chances for the *Ace*, the *Deux*, the *Tray*, and the *Quatre* to come up is $\overline{p+1}|^n - 4 \times p^n + 6 \times \overline{p-1}|^n - 4 \times \overline{p-2}|^n + \overline{p-3}|^n$; the continuation of which proceſs is manifeſt.

Wherefore if all the Powers $\overline{p+1}|^n$; p^n, $\overline{p-1}|^n$, $\overline{p-2}|^n$, $\overline{p-3}|^n$, &c. with Signs alternately poſitive and negative be written in order, and to thoſe Powers there be prefixed the reſpective Coefficients of a Binomial raiſed to the Power f, expreſſing the number of Faces required to come up; the Sum of all thoſe Terms will be the Numerator of the Expectation of A, of which the Denominator will be $\overline{p+1}|^n$.

EXAMPLE I.

Let Six be the number of Faces in the Die, and let A undertake in eight Caſts to fling both an *Ace* and a *Deux*, without any regard to order: then his Expectation will be $\frac{6^8 - 2 \times 5^8 + 4^8}{6^8} = \frac{964702}{1680216} = \frac{4}{7}$ nearly.

EXAM-

EXAMPLE 2.

Let *A* undertake with a common Die to fling all the Faces in 12 Caſts, then his Expectation will be found to be

$$\frac{6^{12} - 6 \times 5^{12} + 15 \times 4^{12} - 20 \times 3^{12} + 15 \times 2^{12} - 6 \times 1^{12} + 1 \times 0^{12}}{6^{12}} = \frac{10}{23}$$

nearly.

EXAMPLE 3.

If *A* with a Die of 36 Faces undertake to fling two given Faces in 43 Caſts; or which is the ſame thing, if with two common Dice he undertake in 43 Caſts to fling two Aces at one time, and two Sixes at another time; his Expectation will be $\frac{36^{43} - 2 \times 35^{43} + 34^{43}}{30^{43}} = \frac{49}{100}$ nearly.

N. B. The parts which compoſe theſe Expectations are eaſily obtained by the help of a Table of Logarithms.

PROBLEM XL.

To find in how many Trials it will be probable that A *with a Die of any given number of Faces ſhall throw any propoſed number of them.*

SOLUTION.

Let $p + 1$ be the number of Faces in the Die, and f the number of Faces which are to be thrown: Divide the Logarithm of $\frac{1}{1 - \sqrt[f]{\frac{1}{2}}}$ by the Logarithm of $\frac{p+1}{p}$, and the Quotient will expreſs the number of Trials requiſite to make it as probable that the propoſed Faces may be thrown as not.

DEMONSTRATION.

Suppoſe Six to be the number of Faces that are to be thrown, and n the number of Trials, then by what has been demonſtrated in the preceding Problem the Expectation of *A* will be

$$\frac{\overline{p+1}^{n} - 6 \times p^{n} + 15 \times \overline{p-1}^{n} + 20 \times \overline{p-2}^{n} - 15 \times \overline{p-3}^{n} + 6 \times \overline{p-4}^{n} + \overline{p-5}^{n}}{\overline{p+1}^{n}}$$

Let it be ſuppoſed that the Terms, $p+1$, p, $p-1$, $p-2$, &c. are in geometric Progreſſion, (which Suppoſition will very little err from the truth, eſpecially if the proportion of p to 1, be not very ſmall.) Let now r be written inſtead of $\frac{p+1}{p}$, and then the Expectation of A will be changed into $1-\frac{6}{r^n}+\frac{15}{r^{2n}}-\frac{20}{r^{3n}}+\frac{15}{r^{4n}}-\frac{6}{r^{5n}}+\frac{1}{r^{6n}}$, or $\overline{1-\frac{1}{r^n}}\Big|^{6}$. But this Expectation of A ought to be made equal to $\frac{1}{2}$, ſince by Suppoſition he has an equal Chance to win or loſe, hence will ariſe the Equation $\overline{1-\frac{1}{r^n}}\Big|^{6}=\frac{1}{2}$ or $r^n=\frac{1}{1-\sqrt[6]{\frac{1}{2}}}$, from which it may be concluded that $n\,\mathrm{Log.}\,r$, or $n\times\mathrm{Log.}\,\frac{p+1}{p}=\log.\,\frac{1}{1-\sqrt[6]{\frac{1}{2}}}$, and conſequently that n is equal to the Logarithm of $\frac{1}{1-\sqrt[6]{\frac{1}{2}}}$, divided by the Logarithm of $\frac{p+1}{p}$. And the ſame demonſtration will hold in any other Caſe.

EXAMPLE I.

To find in how many Trials A may with equal Chance undertake to throw all the Faces of a common Die.

The Logarithm of $\frac{1}{1-\sqrt[6]{\frac{1}{2}}}=0.9621753$; the Logarithm of $\frac{p+1}{p}$ or $\frac{6}{5}=0.0791812$: wherefore $n=\frac{0.9621753}{0.0791812}\doteq 12+$. From hence it may be concluded, that in 12 Caſts A has the worſt of the Lay, and in 13 the beſt of it.

EXAMPLE 2.

To find in how many Trials A may with equal Chance with a Die of thirty-ſix Faces undertake to throw ſix determinate Faces; or, in how many Trials he may with a pair of common Dice undertake to throw all the Doublets.

The Logarithm of $\frac{1}{1-\sqrt[6]{\frac{1}{2}}}$ being 0.9621753, and the Logarithm of $\frac{p+1}{p}$ or $\frac{36}{35}$ being 0.0122345; it follows that the number of Caſts requiſite to that effect is $\frac{0.9621753}{0.0122345}$, or 79 nearly.

But if it were the Law of the Play, that the Doublets muſt be thrown in a given order, and that any Doublet happening to be thrown out of its Turn ſhould go for nothing; then the throwing of the ſix Doublets would be like the throwing of the two Aces ſix times: to produce which, the number of Caſts requiſite would be found by multiplying 35 by 5.668, as appears from the Table annexed to our vth Problem; and conſequently would be about 198.

N. B. The Fraction $\frac{1}{1-\sqrt[f]{\frac{1}{2}}}$, may be reduced to another form *viz.* $\frac{\sqrt[f]{2}}{\sqrt[f]{2}-1}$; which will facilitate the taking of its Logarithm.

PROBLEM XLI.

Suppoſing a regular Priſm having a *Faces marked* I, b *Faces marked* II, c *Faces marked* III, d *Faces marked* IV, &c. *what is the Probability that in a certain number of throws* n, *ſome of the Faces marked* I *will be thrown, as alſo ſome of the Faces marked* II?

SOLUTION.

Make $a+b+c+d$, &c. $=s$, then the Probability required will be expreſſed by $\frac{s^n - \overline{s-a}|^n + \overline{s-a-b}|^n - \overline{s-b}|^n}{s^n}$; the Demonſtration of which flowing naturally from the Method of arguing employed in the xxxixth Problem, there can be no difficulty about it.

EXAMPLE.

Suppoſe it be required to find the Probability of throwing in 8 throws the two Chances v and vi, with a pair of common Dice.

The number of all the Chances upon two Dice being 36, whereof 4 belong to the Chance v, and 5 to the Chance vi; it follows that s ought to be interpreted by 36, a by 4, and b by 5: which being done, the Probability required will be expreſſed by

36^8

$\dfrac{36^8 - 32^8 + 27^8 \atop - 31^8}{36^8}$, which by help of a Table of Log. will be found thus:

$\frac{36^8}{36^8} = 1$, $\frac{32^8}{36^8} = 0.38975$, $\frac{31^8}{36^8} = 0.30176$, $\frac{27^8}{36^8} = 0.10012$, but $1 - 0.38975 - 0.30176 + 0{,}10012 = 0.40861$, and this being ſubtracted from Unity, there remains 0.59139, and therefore the Odds againſt throwing V and VI in 8 throws are 59139 to 40861, that is about 13 to 9.

But if it be required, that ſome of the Faces marked I, ſome of the Faces marked II, and ſome of the Faces marked III, be thrown, the Probability of throwing thoſe Chances in a given number of throws n will be expreſſed by

$$\frac{\begin{array}{l} s^n - \overline{s-a}|^n + \overline{s-a-b}|^n - \overline{s-a-b-c}|^n \\ \quad - \overline{s-b}|^n + \overline{s-a-c}|^n \\ \quad - \overline{s-c}|^n + \overline{s-b-c}|^n \end{array}}{s^n}$$

And if the Faces marked IV are farther required to be thrown, the Probability of it will be expreſſed by

$$\frac{\begin{array}{lllll} s^n & - \overline{s-a}|^n & + \overline{s-a-b}|^n & - \overline{s-a-b-c}|^n & + \overline{s-a-b-c-d}|^n \\ & - \overline{s-b}|^n & + \overline{s-a-c}|^n & - \overline{s-a-b-d}|^n & \\ & - \overline{s-c}|^n & + \overline{s-a-d}|^n & - \overline{s-a-c-d}|^n & \\ & - \overline{s-d}|^n & + \overline{s-b-c}|^n & - \overline{s-b-c-d}|^n & \\ & & + \overline{s-b-d}|^n & & \\ & & + \overline{s-c-d}|^n & & \end{array}}{s^n}$$

Now the order of the preceding Solutions being manifeſt, it will be eaſy by bare inſpection to continue them as far as there is occaſion.

PROBLEM XLII.

If A *obliges himſelf in a certain number of throws* n *with a pair of common Dice not only to throw the Chances* V *and* VI, *but* V *before* VI; *with this reſtriction, that if he happens to throw* VI *before* V, *he does not indeed loſe his wager, but is to proceed as if nothing had been done, ſtill deducting ſo many throws as have been vain from the number of throws which he had at firſt given him; to find the Probability of his winning.*

SOLU-

SOLUTION.

Let the number of Chances which there are for throwing v be called *a*, the number of Chances for throwing vi, *b*; the number of all the Chances upon two Dice *ſ*, and the number of throws that *A* takes $= n$. This being ſuppoſed,

1°. If *A* throws v the firſt throw, of which the Probability is $\frac{a}{ſ}$, he has nothing more to do than to throw vi in $n-1$ times, of which the Probability is $1 - \frac{\overline{ſ-b}^{n-1}}{ſ^{n-1}}$, and therefore the Probability of throwing v the firſt time, and throwing afterwards vi in $n-1$ times is $\frac{a}{ſ} \times \overline{1 - \frac{\overline{ſ-b}^{n-1}}{ſ^{n-1}}}$.

2°, If *A* miſſes v the firſt time, and throws it the ſecond, of which the Probability is $\frac{ſ-a}{ſ} \times \frac{a}{ſ}$, then he is afterwards to throw vi in $n-2$ times, of which the Probability being $1 - \frac{\overline{ſ-b}^{n-2}}{ſ^{n-2}}$ it follows that the Probability of miſſing v the firſt time, throwing it the ſecond, and afterwards throwing vi, will be $\frac{ſ-a}{ſ} \times \frac{a}{ſ} \times 1 - \frac{\overline{ſ-b}^{n-2}}{ſ^{n-2}}$.

3°. If *A* miſſes v the two firſt times, and throws it the third, then he is afterwards to throw vi in $n-3$ times, the Probability of all which is $\frac{\overline{ſ-a}^2}{ſſ} \times \frac{a}{ſ} \times 1 - \frac{\overline{ſ-b}^{n-3}}{ſ^{n-3}}$; and ſo on. Now all this added together conſtitutes two geometric Progreſſions, the number of whoſe Terms in each is $n-1$.

Wherefore the Sum of the whole will be

$$\frac{ſ^{n-1} - \overline{ſ-a}^{n-1}}{ſ^{n-1}} - \frac{aſ-ab}{a-b} \times \frac{\overline{ſ-b}^{n-1} - \overline{ſ-a}^{n-1}}{ſ^{n}} : \text{and}$$

if *a* and *b* are equal, then the ſecond part will be reduced to $-\overline{n-1} \times a \times \overline{s-a}^{n-1} \times \frac{1}{ſn}$.

Now for the application of this to numbers; *a* in the Caſe propoſed is $= 4$, $b = 5$, $s = 36$. Let *n* be $= 12$, and the Probability required will be found to be 0.44834, which being ſubtracted from unity the remainder will be 0.55166, and therefore the Odds againſt *A* are 55166 to 44834, that is nearly as 21 to 17.

But if the conditions of the Play were that *A* in 12 times ſhould throw both V and VI, and that VI ſhould come up before V, the Odds againſt *A* would not be ſo great; being only 54038 to 45962, that is nearly as 20 to 17.

It would not be difficult after what we have ſaid, tho' perhaps a little laborious, to extend theſe kinds of Solutions to any number of Chances given.

PROBLEM XLIII.

Any number of Chances being given, to find the Probability of their being produced in a given order, without any limitation of the number of times in which they are to be produced.

SOLUTION.

1°. Let the Chances be a and b, and let it be required to produce them in the order a, b.

The Probability of producing a before b is $\frac{a}{a+b}$, which being ſuppoſed to have happened, b muſt be produced of neceſſity; and therefore the Probability of producing the Chances a and b in the given order a, b, is $\frac{a}{a+b}$.

2°. Let the Chances given be a, b, c, and let it be required to produce them in the order in which they are written; then the Probability of producing a before b or c is $\frac{a}{a+b+c}$; which being ſuppoſed, the Probability of producing b before c is by the preceding caſe $\frac{b}{b+c}$; after which c muſt neceſſarily be produced, and therefore the Probability of this caſe is $\frac{a}{a+b+c} \times \frac{b}{b+c}$.

3°. Let the Chances be a, b, c, d, and let it be required to produce them in the order in which they are written; then the Probability of producing a before all the reſt is $\frac{a}{a+b+c+d}$; which being ſuppoſed, the Probability of prodocing b before all the remaining is $\frac{b}{b+c+d}$; which being ſuppoſed, the Probability of producing c before d is $\frac{c}{c+d}$. And therefore the Probability of the whole is $\frac{a}{a+b+c+d} \times \frac{b}{b+c+d} \times \frac{c}{c+d}$; and in the ſame manner may theſe Theorems be continued *in infinitum*.

And therefore if it was propoſed to find the Probability of throwing with a pair of common Dice the Chances IV, V, VI, VIII, IX, X before VII; let the Chances be called reſpectively $a, b, c, d, e, f,$ and m, then the Probability of throwing them in the order they are writ in will be

$$\frac{a}{a+b+c+d+e+f+m} \times \frac{b}{b+c+d+e+f+m} \times \frac{c}{c+d+e+f+m} \times \frac{d}{d+e+f+m} \times \frac{e}{e+f+m} \times \frac{f}{f+m}.$$

But as the order in which they may be thrown is not the thing particularly required here, except that the Chances m are to be thrown the laſt; ſo it is plain that there will be as many different parts like the preceding as the poſition of the 6 Letters $a, b, c, d, e, f,$ may be varied, which being 720 different ways, it follows, that in order to have a compleat Solution of this Queſtion, there muſt be 720 different parts like the preceding to be added together.

However the Chances IV and X, V and IX, VI and VIII being reſpectively the ſame, thoſe 720 might be reduced to 90, which being added together, and the Sum multiplied by 8, we ſhould have the Probability required.

Still thoſe Operations would be laborious, for which reaſon it will be ſufficient to have an approximation, by ſuppoſing that all the Chances $a, b, c, d, e, f,$ that is, 3, 4, 5, 5, 4, 3 are equal to the mean Chance 4, which will make it that the Probability required will be expreſſed by

$$\frac{6b}{6b+m} \times \frac{5b}{5b+m} \times \frac{4b}{4b+m} \times \frac{3b}{3b+m} \times \frac{2b}{2b+m} \times \frac{b}{b+m} \text{ or}$$
$$\frac{24}{30} \times \frac{20}{26} \times \frac{16}{22} \times \frac{12}{18} \times \frac{8}{14} \times \frac{4}{10} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 4 \cdot 4}{1 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13} = \frac{1024}{15015};$$

and therefore the Odds againſt throwing the Chances IV, V, VI, VIII, IX, X before VII are about 13991 to 1024, or nearly 41 to 3.

But the Solution might be made ſtill more exact, if inſtead of taking 4 for the mean Chance, we find the ſeveral Probabilities of throwing all the Chances before VII, and take the ſixth part of the Sum for the mean Probability; thus becauſe the ſeveral Probabilities of throwing all the Chances before VII are reſpectively $\frac{3}{9}$, $\frac{4}{10}$, $\frac{5}{165}$, $\frac{5}{11}$, $\frac{4}{10}$, $\frac{3}{9}$, the Sum of all which is $\frac{392}{165}$, if we divide the whole by 6, the Quotient will be $\frac{392}{990}$ or $\frac{50}{149}$

nearly, and this being ſuppoſed $= \frac{z}{z+6}$ wherein z repreſents the mean Chance, we ſhall find $z = 3\,\frac{14}{15}$. And therefore the Probability of throwing all the Chances before VII, will be found to be $\frac{354}{444} \times \frac{295}{385} \times \frac{236}{326} \times \frac{177}{267} \times \frac{118}{208} \times \frac{59}{149} = 0.065708$ nearly, which being ſubtracted from Unity, the remaining is 0.934292, and therefore the Odds againſt throwing all the Chances before VII are 934292 to 65708, that is about $14\,\frac{1}{5}$ to 1.

But if it was farther required not only to throw all the Chances before VII, but alſo to do it in a certain number of times aſſigned, the Problem might eaſily be ſolved by imagining a mean Chance.

PROBLEM XLIV.

If A, B, C *play together on the following conditions; Firſt that they ſhall each of them ſtake* 1

Secondly that A *and* B *ſhall begin the Play; Thirdly, that the Loſer ſhall yield his place to the third Man, which is conſtantly to be obſerved afterwards; Fourthly, that the Loſer ſhall be fined a certain Sum* p, *which is to ſerve to increaſe the common Stock; Laſtly, that he ſhall have the whole Sum depoſited at firſt, and increaſed by the ſeveral Fines, who ſhall firſt beat the other two ſucceſſively: 'Tis demanded what is the Advantage or Diſadvantage of* A *and* B, *whom we ſuppoſe to begin the Play.*

SOLUTION.

Let BA ſignify that *B* beats *A*, and AC that *A* beats *C*, and ſo let always the firſt Letter denote the Winner, and the ſecond the Loſer.

Let us ſuppoſe that *B* beats *A* the firſt time; then let us inquire what the Probability is that the Set ſhall be ended in any number of Games, and alſo what is the Probability which each Gameſter has of winning the Set in that given number of Games.

Firſt, If the Set be ended in two Games, *B* muſt neceſſarily be the winner, for by Hypotheſis he wins the firſt time; which may be expreſſed by BA, BC.

Secondly, If the Set be ended in three Games, *C* muſt be the winner, as appears by the following Scheme, *viz.* BA, CB, CA.

Thirdly, If the Set be ended in four Games, *A* muſt be the winner, as appears by the Scheme BA, CB, AC, AB.

Fourthly, If the Set be ended in five Games, *B* muſt be the winner, which is thus expreſſed, BA, CB, AC, BA, BC.

Fifthly, If the Set be ended in ſix Games, *C* muſt be the winner, as appears ſtill by the following proceſs, thus, BA, CB, AC, BA, CB, CA.

And this proceſs recurring continually in the ſame order needs not be proſecuted any farther.

Now the Probability that the firſt Scheme ſhall take place is $\frac{1}{2}$, in conſequence of the Suppoſition made that *B* beats *A* the firſt time; it being an equal Chance whether *B* beat *C*, or *C* beat *B*.

And the Probability that the ſecond Scheme ſhall take place is $\frac{1}{4}$: for the Probability of *C*'s beating *B* is $\frac{1}{2}$, and that being ſuppoſed, the Probability of his beating *A* will alſo be $\frac{1}{2}$; wherefore the Probability of *C*'s beating *B*, and then *A*, will be $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

And from the ſame conſideration, the Probability that the third Scheme ſhall take place is $\frac{1}{8}$: and ſo on.

Hence it will be eaſy to compoſe a Table of the Probabilities which *B*, *C*, *A* have of winning the Set in any given number of Games; and alſo of their Expectations: which Expectations are the Probabilities of winning multiplied by the common Stock depoſited at firſt, and increaſed ſucceſſively by the ſeveral Fines.

TABLE *of the Probabilities*, &c.

	B	*C*	*A*
2	$\frac{1}{2}\times\overline{3+2p}$	----------	----------
3	----------	$\frac{1}{4}\times\overline{3+3p}$	----------
4	----------	----------	$\frac{1}{8}\times\overline{3+4p}$
5	$\frac{1}{16}\times\overline{3+5p}$	----------	----------
6	----------	$\frac{1}{32}\times\overline{3+6p}$	----------
7	----------	----------	$\frac{1}{64}\times\overline{3+7p}$
8	$\frac{1}{128}\times\overline{3+8p}$	----------	----------
9	----------	$\frac{1}{256}\times\overline{3+9p}$	----------
10 &c.	----------	----------	$\frac{1}{512}\times\overline{3+10p}$

Now the ſeveral Expectations of *B*, *C*, *A* may be ſummed up by the following Lemma.

LEMMA.

$\frac{n}{b}+\frac{n+d}{bb}+\frac{n+2d}{b^3}+\frac{n+3d}{b^4}+\frac{n+4d}{b^5}$, &c. *in infinitum*, is equal to $\frac{n}{b-1}+\frac{d}{\overline{b-1}|^2}$.

Let the Expectations of *B* be divided into two Series, *viz.*

$$\frac{3}{2}+\frac{3}{16}+\frac{3}{128}+\frac{3}{1024}, \text{ \&c.}$$

$$+\frac{2p}{2}+\frac{5p}{16}+\frac{8p}{128}+\frac{11p}{1024}, \text{ \&c.}$$

The firſt Series conſtituting a Geometric Progreſſion continually decreaſing, its Sum by the known Rules will be found to be $\frac{12}{7}$.

The ſecond Series may be reduced to the form of the Series in our Lemma, and may be thus expreſſed

$\frac{p}{2} \times \overline{\frac{2}{1} + \frac{5}{8} + \frac{8}{8^2} + \frac{11}{8^3} + \frac{14}{8^4}}$, &c. wherefore dividing the whole by $\frac{p}{2}$, and laying aſide the firſt term 2, we ſhall have the Series $\frac{5}{8} + \frac{8}{8^2} + \frac{11}{8^3} + \frac{14}{8^4}$, &c. which has the ſame form as the Series of the Lemma, and may be compared with it: let therefore n be made $= 5$, $d = 3$, and $b = 8$, and the Sum of the Series will be $\frac{5}{7} + \frac{3}{49}$ or $\frac{38}{49}$; to this adding the firſt Term 2 which had been laid aſide, the new Sum will be $\frac{136}{49}$, and that being multiplied by $\frac{p}{2}$ whereby it had been divided, the product will be $\frac{68}{49}p$, which is the Sum of the ſecond Series expreſſing the Expectation of B: from whence it may be concluded that all the Expectations of B contained in both the abovementioned Series will be equal to $\frac{12}{7} + \frac{68}{49}p$.

And by the help of the ſame *Lemma*, it will be found that all the Expectations of C will be equal to $\frac{6}{7} + \frac{48}{49}p$.

It will be alſo found that all the Expectations of A will be equal to $\frac{3}{7} + \frac{31}{49}p$.

We have hitherto determined the ſeveral Expectations of the Gameſters upon the Sum by them depoſited at firſt, and alſo upon the Fines by which the common Stock is increaſed: it now remains to eſtimate the ſeveral Riſks of their being fined; that is to ſay, the Sum of the Probabilities of their being fined multiplied by the reſpective Values of the Fines.

Now after the Suppoſition made of A's being beat the firſt time, by which he is obliged to lay down his Fine p, B and C have an equal Chance of being fined after the ſecond Game; which makes the Riſk of each to be $= \frac{1}{2}p$, as appears by the following Scheme.

$$\frac{BA}{CB} \text{ or } \frac{BA}{BC}$$

In like manner, it will be found, that C and A have one Chance in four, for their being fined after the third Game, and conſequently that the Riſk of each is $\frac{1}{4}p$, according to the following Scheme.

$$\begin{array}{ccc} \underline{BA} & & \underline{BA} \\ CB & \text{or} & CB \\ AC & & CA \end{array}$$

And by the like Proceſs, it will be found that the Riſk of *B* and *C* after the fourth Game is $\frac{1}{8}p$.

Hence it will be eaſy to compoſe the following Table, which expreſſes the Riſks of each Gameſter.

TABLE *of Riſks.*

	B	*C*	*A*
2	$\frac{1}{2}p$	$\frac{1}{2}p$	-----
3	-----	$\frac{1}{4}p$	$\frac{1}{4}p$
4	$\frac{1}{8}p$	-----	$\frac{1}{8}p$
5	$\frac{1}{16}p$	$\frac{1}{16}p$	-----
6	-----	$\frac{1}{32}p$	$\frac{1}{32}p$
7	$\frac{1}{64}p$	-----	$\frac{1}{64}p$
8	$\frac{1}{128}p$	$\frac{1}{128}p$	-----
9	-----	$\frac{1}{256}p$	$\frac{1}{256}p$
&c.			

In the Column belonging to *B*, if the vacant places were filled up by interpolating the Terms $\frac{1}{4}p$, $\frac{1}{32}p$, $\frac{1}{256}p$, &c. the Sum of the Riſks of *B* would compoſe one uninterrupted geometric Progreſſion, the Sum of whoſe Terms would be $= p$; but the Terms interpolated conſtitute a geometric Progreſſion whoſe Sum is $= \frac{2}{7}p$: wherefore if from p there be ſubtracted $\frac{2}{7}p$, there will remain $\frac{5}{7}p$ for the Sum of the Riſks of *B*.

In like manner it will be found that the Sum of the Riſks of *C* will be $= \frac{6}{7}p$.

And the Sum of the Riſks of *A*, after his being fined the firſt time, will be $\frac{3}{7}p$.

Now if from the ſeveral Expectations of the Gameſters, there be ſubtracted each Man's Stake, and alſo the Sum of his Riſks, there will remain the clear Gain or Loſs of each of them.

Wherefore, from the Expectations of $B = \frac{12}{7} + \frac{68}{49}p$.
Subtracting *firſt* his Stake $= 1$
Then the Sum of his Riſks $= \frac{5}{7}p$.
There remains the clear Gain of $B = \frac{5}{7} + \frac{33}{49}p$.

Likewiſe from the Expectations of $C = \frac{6}{7} + \frac{48}{49}p$.
Subtracting *firſt* his Stake $= 1$
Then the Sum of his Riſks $= \frac{6}{7}p$.
There remains the clear Gain of $C = -\frac{1}{7} + \frac{6}{49}p$.

In like manner, from the Expectations of $A = \frac{3}{7} + \frac{31}{49}p$.
Subtracting, *firſt* his Stake $= 1$
Secondly, the Sum of his Riſks $= \frac{3}{7}p$.
Laſtly, the Fine p due to the Stock by the Loſs of the firſt Game $= p$.
There remains the clear Gain of $A = -\frac{4}{7} - \frac{39}{49}p$.

But we had ſuppoſed, that in the beginning of the Play *A* was beaten; whereas *A* had the ſame Chance to beat *B*, as *B* had to beat him: wherefore dividing the Sum of the Gains of *B* and *A* into two equal parts, each Part will be $\frac{1}{14} - \frac{3}{49}p$, which conſequently muſt be reputed to be the Gain of each of them.

COROLLARY I.

The Gain of *C* being $-\frac{1}{7} + \frac{6}{49}p$, let that be made $= 0$, then p will be found to be $= \frac{7}{6}$. If therefore the Fine has the ſame proportion to each Man's Stake as 7 has to 6, the Gameſters play all upon equal terms: But if the Fine bears a leſs proportion to the Stake than 7 to 6, *C* has the diſadvantage: thus ſuppoſing $p = 1$, his Loſs would be $\frac{1}{49}$, but if it bears a greater proportion to the Stake than 7 to 6, *C* has the advantage.

COROLLARY 2.

If the common Stake were conſtant, that is if there were no Fines, then the Probabilities of winning would be proportional to the Expectations; wherefore ſuppoſing $p = 0$, the Expectations after the firſt Game would be $\frac{12}{7}$, $\frac{6}{7}$, $\frac{3}{7}$, whereof the firſt belongs to *B*, the ſecond to *C*, and the third to *A*: and therefore dividing the Sum of the Probabilities belonging to *B* and *A* into two equal parts, it will follow that the Probabilities of winning would be proportional to the numbers 5, 4, 5, and therefore it is five to two before the Play begins that either *A* or *B* win the Set, or five to four that one of them that ſhall be fixed upon wins it.

COROLLARY 3.

Hence likewiſe if three Gameſters *A*, *B*, *C*, are engaged in a *Poule*, and have not time to play it out; but agree to divide (*S*) the Sum of the Stake and Fines, in proportion to their reſpective Chances: $\frac{4}{7}S$ will be the Share of *B*, whom we ſuppoſe to have got one Game; $\frac{2}{7}S$ that of *C*, who ſhould next come in; and $\frac{1}{7}S$ the Share of *A* who was laſt beat. For, as they agree to give over playing, all conſideration of the ſubſequent Fines *p* is now ſet aſide, and the Caſe comes to that of the firſt part of *Corol.* 2.

Or the ſame thing may be ſhortly demonſtrated as follows.

Put $S = 1$, and the Share of $A = z$. Then *B* playing with *C* has an equal Chance for the whole Stake *S*, and for being reduced to the preſent Expectation of *A*; that is, *B*'s Expectation is $\frac{1+z}{2}$. *C* has an equal chance for 0, and for *B*'s preſent Expectation; that is, *C*'s Expectation is $\frac{0 + \frac{1}{2} \times \overline{1+z}}{2} = \frac{1+z}{4}$. But the Sum of the three Expectations $z + \frac{1}{2} \times \overline{1+z} + \frac{1}{4} \times \overline{1+z} = S = 1$; or $z + \frac{3}{4}z \left(= \frac{7}{4}z\right) = \frac{1}{4}$: and $z = \frac{1}{7}$, which is *A*'s Share; thoſe of *B* and *C* being $\frac{1}{2} \times \overline{1 + \frac{1}{7}}$, and $\frac{1}{4} \times \overline{1 + \frac{1}{7}}$; or $\frac{4}{7}$ and $\frac{2}{7}$, reſpectively.

PRO-

PROBLEM XLV.

If four Gameſters play on the conditions of the foregoing Problem, and he be to be reputed the Winner who beats the other three ſucceſſively, what is the Advantage of A *and* B *whom we ſuppoſe to begin the Play?*

SOLUTION.

Let BA denote as in the preceding Problem that *B* beats *A*, and AC that *A* beats *C*; and univerſally, let the firſt Letter always denote the Winner, and the ſecond the Loſer.

Let it be alſo ſuppoſed that *B* beats *A* the firſt time: then let it be inquired what is the Probability that the Play ſhall be ended in any number of Games; as alſo what is the Probability which each Gameſter has of winning the Set in that given number of Games.

Firſt, If the Set be ended in three Games, *B* muſt neceſſarily be the Winner; ſince by hypotheſis he beats *A* the firſt Game, which is expreſſed as follows:

1	BA
2	$\overline{\text{BC}}$
3	BD

Secondly, If the Set be ended in four Games, *C* muſt be the winner; as it thus appears.

1	BA
2	$\overline{\text{CB}}$
3	CD
4	CA

Thirdly, If the Set be ended in five Games, *D* will be the Winner; for which he has two Chances, as it appears by the following Scheme.

1	BA		BA
2	$\overline{\text{CB}}$		$\overline{\text{BC}}$
3	DC	or	DB
4	DA		DA
5	DB		DC

Fourthly, If the Set be ended in ſix Games, *A* will be the Winner; and he has three Chances for it, which are thus collected.

1	BA	BA	BA
2	$\overline{CB}$	$\overline{CB}$	$\overline{BC}$
3	DC	CD	DB
4	AD	AC	AD
5	AB	AB	AC
6	AC	AD	AB

Fifthly, If the Set be ended in ſeven Games, then *B* will have three Chances to be the Winner, and *C* will have two, thus;

1	BA	BA	BA	BA	BA
2	$\overline{CB}$	$\overline{CB}$	$\overline{CB}$	$\overline{BC}$	$\overline{BC}$
3	DC	DC	CD	DB	DB
4	AD	DA	AC	AD	DA
5	BA	BD	BA	CA	CD
6	BC	BC	BD	CB	CB
7	BD	BA	BC	CD	CA

Sixthly, If the Set be divided in eight Games, then *D* will have two Chances to be the Winner, *C* will have three, and *B* alſo three, thus;

1	BA	BA	BA	BA	BA	BA	BA	BA
2	$\overline{CB}$	$\overline{CB}$	$\overline{CB}$	$\overline{CB}$	$\overline{CB}$	$\overline{BC}$	$\overline{BC}$	$\overline{BC}$
3	DC	DC	DC	CD	CD	DB	DB	DB
4	AD	AD	DA	AC	AC	AD	AD	DA
5	BA	AB	BD	BA	AB	CA	AC	CD
6	CB	CA	CB	DB	DA	BC	BA	BC
7	CD	CD	CA	DC	DC	BD	BD	BA
8	CA	CB	CD	DA	DB	BA	BC	BD

Let now the Letters by which the Winners are denoted be written in order, prefixing to them the numbers which expreſs their ſeveral Chances for winning; in this manner.

3	$1B$
4	$1C$
5	$2D$
6	$3A$
7	$3B + 2C$
8	$3C + 2D + 3B$
9	$3D + 2A + 3C + 3D + 2A$
10	$3A + 2B + 3D + 3A + 2B + 2C + 3A + 3D$
&c.	

Then carrying this Table a littler farther, and examining the Formation of theſe Letters, it will appear; *Firſt*, that the Letter B is always found ſo many times in any Rank, as the Letter A is found in the two preceding Ranks: *Secondly*, that C is found ſo many times in any Rank as B is found in the preceding Rank, and D in the Rank before that. *Thirdly*, that D is found ſo many times in any Rank, as C is found in the preceding, and B in the Rank before that: And, *Fourthly*, that A is found ſo many times in any Rank as D is found in the preceding Rank, and C in the Rank before that.

From all which it may be concluded, that the Probability which the Gameſter *B* has of winning the Set in any number of Games, is $\frac{1}{2}$ of the Probability which *A* has of winning it one Game ſooner, together with $\frac{1}{4}$ of the Probability which *A* has of winning it two Games ſooner.

The Probability which *C* has of winning the Set in any given number of Games, is $\frac{1}{2}$ of the Probability which *B* has of winning it one Game ſooner, together with $\frac{1}{4}$ of the Probability which *D* has of winning it two Games ſooner.

The Probability which *D* has of winning the Set in any number of Games is $\frac{1}{2}$ the Probability which *C* has of winning it one Game ſooner, and alſo $\frac{1}{4}$ of the Probability which *B* has of winning it two Games ſooner.

The Probability which *A* has of winning the Set in any number of Games is $\frac{1}{2}$ of the Probability which *D* has of winning it one Game ſooner, and alſo $\frac{1}{4}$ of the Probability which *C* has of winning it two Games ſooner.

Theſe things being obſerved, it will be eaſy to compoſe a Table of the Probabilities which *B*, *C*, *D*, *A* have of winning the Set in any number of Games, as alſo of their Expectations, which will be as follows:

		B	*C*	*D*	*A*
I	3	$\frac{1}{4}\times\overline{4+3p}$	---------	---------	---------
II	4	---------	$\frac{1}{8}\times\overline{4+4p}$	---------	---------
III	5	---------	---------	$\frac{2}{16}\times\overline{4+5p}$	---------
IV	6	---------	---------	---------	$\frac{3}{32}\times\overline{4+6p}$
V	7	$\frac{3}{64}\times\overline{4+7p}$	$\frac{2}{64}\times\overline{4+7p}$	---------	---------
VI	8	$\frac{3}{128}\times\overline{4+8p}$	$\frac{3}{128}\times\overline{4+8p}$	$\frac{2}{128}\times\overline{4+8p}$	---------
VII	9	---------	$\frac{3}{256}\times\overline{4+9p}$	$\frac{6}{256}\times\overline{4+9p}$	$\frac{4}{256}\times\overline{4+9p}$
VIII	10	$\frac{4}{512}\times\overline{4+10p}$	$\frac{2}{512}\times\overline{4+10p}$	$\frac{6}{512}\times\overline{4+10p}$	$\frac{9}{512}\times\overline{4+10p}$
IX	11	$\frac{13}{1024}\times\overline{4+11p}$	$\frac{10}{1024}\times\overline{4+11p}$	$\frac{2}{1024}\times\overline{4+11p}$	$\frac{9}{1024}\times\overline{4+11p}$
X	12	$\frac{18}{2048}\times\overline{4+12p}$	$\frac{19}{2048}\times\overline{4+12p}$	$\frac{14}{2048}\times\overline{4+12p}$	$\frac{4}{2048}\times\overline{4+12p}$
&c.	&c				

The Terms whereof each Column of this Table is compoſed, being not eaſily ſummable by any of the known Methods, it will be convenient, in order to find their Sums to uſe the following *Analyſis.*

Let $B' + B'' + B''' + B^{IV} + B^{V} + B^{VI}$, &c. repreſent the reſpective Probabilities which *B* has of winning the Set, in any number of Games anſwering to 3, 4, 5, 6, 7, 8, &c. and let the Sum of theſe Probabilities *in infinitum* be ſuppoſed $= y$.

In the ſame manner, let $C' + C'' + C''' + C^{V} + C^{V} + C^{VI}$, &c. repreſent the Probabilities which *C* has of winning, which ſuppoſe $= z$.

Let the Probabilities which *D* has of winning be repreſented by $D' + D'' + D''' + D^{IV} + D^{V} + D^{VI}$, &c. which ſuppoſe $= v$.

Laſtly, Let the Probabilities which *A* has of winning be repreſented by $A' + A'' + A''' + A^{IV} + A^{V} + A^{VI}$, &c. which ſuppoſe $= x$.

Now from the Obſervations ſet down before in the Table of Probabilities, it will follow, that

B'

$$\begin{aligned}
&B' = B' \\
&B'' = B'' \\
&B''' = \tfrac{1}{2}A'' + \tfrac{1}{4}A' \\
&B^{IV} = \tfrac{1}{2}A''' + \tfrac{1}{4}A'' \\
&B^{V} = \tfrac{1}{2}A^{IV} + \tfrac{1}{4}A''' \\
&B^{VI} = \tfrac{1}{2}A^{V} + \tfrac{1}{4}A^{IV} \\
&\&c.
\end{aligned}$$

From which Scheme we may deduce the Equation following, $y = \frac{1}{4} + \frac{3}{4}x$: for the Sum of the Terms in the firſt Column is equal to the Sum of the Terms in the other two. But the Sum of the Terms in the firſt Column is $= y$ by Hypotheſis; wherefore y ought to be made equal to the Sum of the Terms in the other two Columns.

In order to find the Sum of the Terms of the ſecond Column, I argue thus,

$$A' + A'' + A''' + A^{IV} + A^{V} + A^{VI}, \&c. = x \text{ by } \textit{Hypoth.}$$

$$\text{Theref. } A'' + A''' + A^{IV} + A^{V} + A^{VI}, \&c. = x - A'$$

$$\text{and } \tfrac{1}{2}A'' + \tfrac{1}{2}A''' + \tfrac{1}{2}A^{IV} + \tfrac{1}{2}A^{V} + \tfrac{1}{2}A^{VI}, \&c. = \tfrac{1}{2}x - \tfrac{1}{2}A'$$

Then adding $B' + B''$ on both Sides of the laſt Equation, we ſhall have

$$B' + B'' + \tfrac{1}{2}A'' + \tfrac{1}{2}A''' + \tfrac{1}{2}A^{IV} + \tfrac{1}{2}A^{V} + \tfrac{1}{2}A^{VI}, \&c.$$
$$= \tfrac{1}{2}x - \tfrac{1}{2}A' + B' + B''.$$

But $A' = 0$, $B' = \frac{1}{4}$, $B'' = 0$, as appears from the Table: wherefore the Sum of the Terms of the ſecond Column is $= \frac{1}{2}x + \frac{1}{4}$.

The Sum of the Terms of the third Column is $\frac{1}{4}x$ by Hypotheſis; and conſequently the Sum of the Terms in the ſecond and third Columns is $= \frac{3}{4}x + \frac{1}{4}$, from whence it follows that the Equation $y = \frac{1}{4} + \frac{3}{4}x$ had been rightly determined.

And by a reaſoning like the preceding, we ſhall find $z = \frac{1}{2}y + \frac{1}{4}v$, and alſo $v = \frac{1}{2}z + \frac{1}{4}y$, and laſtly $x = \frac{1}{2}v + \frac{1}{4}z$.

Now theſe four Equations being reſolved, it will be found that

$$B^{I} + B^{II} + B^{III} + B^{IV} + B^{V} + B^{VI}, \text{ \&c.} = y = \frac{56}{149}$$
$$C^{I} + C^{II} + C^{III} + C^{IV} + C^{V} + C^{VI}, \text{ \&c.} = z = \frac{36}{149}$$
$$D^{I} + D^{II} + D^{III} + D^{IV} + D^{V} + D^{VI}, \text{ \&c.} = v = \frac{32}{149}$$
$$A^{I} + A^{II} + A^{III} + A^{IV} + A^{V} + A^{VI}, \text{ \&c.} = x = \frac{25}{149}$$

Hitherto we have determined the Probabilities of winning: but in order to find the ſeveral Expectations of the Gameſters, each Term of the Series expreſſing thoſe Probabilities is to be multiplied by the reſpective Terms of the following Series,

$$4 + 3p,\ 4 + 4p,\ 4 + 5p,\ 4 + 6p,\ 4 + 7p,\ 4 + 8p, \text{ \&c.}$$

The firſt part of each product being no more than a Multiplication by 4, the Sums of all the firſt parts of thoſe Products are only the Sums of the Probabilities multiplied by 4; and conſequently are $4y$, $4z$, $4v$, $4x$, or $\frac{224}{149}$, $\frac{144}{149}$, $\frac{128}{149}$, $\frac{100}{149}$, reſpectively.

But to find the Sums of the other parts,

Let $3B^{I}p + 4B^{II}p + 5B^{III}p + 6B^{IV}p$, &c. be $= pt$.
$3C^{I}p + 4C^{I}p + 5C^{III}p + 6C^{IV}p$, &c. be $= ps$.
$3D^{I}p + 4D^{II}p + 5D^{III}p + 6D^{IV}p$, &c. be $= pr$.
$3A^{I}p + 4A^{II}p + 5A^{III}p + 6A^{IV}p$, &c. be $= pq$.

Now ſince
$$3B^{I} = 3B^{I}$$
$$4B^{II} = 4B^{II}$$
$$5B^{III} = \frac{5}{2}A^{II} + \frac{5}{4}A^{I}$$
$$6B^{IV} = \frac{6}{2}A^{III} + \frac{6}{4}A^{II}$$
$$7B^{V} = \frac{7}{2}A^{IV} + \frac{7}{4}A^{III}$$
$$8B^{VI} = \frac{8}{2}A^{V} + \frac{8}{4}A^{IV}$$

it follows that $t = \frac{3}{4} + \frac{3}{4}q + x$; for 1°, the firſt Column is $= t$ by *Hypotheſis.*

2°, $3A^{I} + 4A^{II} + 5A^{III} + 6A^{IV} + 7A^{V}$, &c. $= q$ by *Hypotheſis.*

3°, $A^{I} + A^{II} + A^{III} + A^{IV} + A^{V}$, &c. has been found $= \frac{25}{49}$ $=$ to the value of x.

Where

Wherefore adding theſe two Equations together, we ſhall have

$$4A^{I} + 5A^{II} + 6A^{III} + 7A^{IV} + 8A^{V}, \text{ \&c.} = q + x,$$

or $\frac{4}{2}A^{I} + \frac{5}{2}A^{II} + \frac{6}{2}A^{III} + \frac{7}{2}A^{IV} + \frac{8}{2}A^{V}$, &c. $= \frac{1}{2}q + \frac{1}{2}x$.

But $A^{I} = 0$, therefore there remains ſtill

$$\frac{5}{2}A^{II} + \frac{6}{2}A^{III} + \frac{7}{2}A^{IV} + \frac{8}{2}A^{V}, \text{ \&c.} = \frac{1}{2}q + \frac{1}{2}x.$$

Now the Terms of this laſt Series, together with $3B^{I} + 4B^{II}$, compoſe the ſecond Column: but $3B^{I} = \frac{3}{4}$ and $4B^{II} = 0$, as appears from the Table; conſequently the Sum of the Terms of the ſecond Column is $= \frac{3}{4} + \frac{1}{2}q + \frac{1}{2}x$.

By the ſame Method of proceeding, it will be found that the Sum of the Terms of the third Column is $= \frac{1}{4}q + \frac{1}{2}x$.

From whence it follows, that $y = \frac{3}{4} + \frac{1}{2}q + \frac{1}{2}x + \frac{1}{4}q + \frac{1}{2}x$ or $t = \frac{3}{4} + \frac{3}{4}q + x$.

And by the ſame way of reaſoning, we ſhall find

$s = \frac{1}{2}t + \frac{1}{2}y + \frac{1}{4}r + \frac{1}{2}v$, and alſo
$r = \frac{1}{2}s + \frac{1}{2}z + \frac{1}{4}t + \frac{1}{2}y$, and laſtly
$q = \frac{1}{2}r + \frac{1}{2}v + \frac{1}{4}s + \frac{1}{2}z$.

But for avoiding confuſion, it will be proper to reſtore the values of x, y, z, v, which being done, the Equations will ſtand as follows.

$$t = \frac{3}{4} + \frac{3}{4}q + \frac{25}{149} \text{ or } t = \frac{547}{596} + \frac{3}{4}q.$$
$$s = \frac{44}{149} + \frac{1}{2}t + \frac{1}{4}r.$$
$$r = \frac{46}{149} + \frac{1}{2}s + \frac{1}{4}t.$$
$$q = \frac{34}{149} + \frac{1}{2}r + \frac{1}{4}s.$$

Now the foregoing Equations being ſolved, it will be found that $t = \frac{45536}{22201}$, $s = \frac{38724}{22201}$, $r = \frac{37600}{22201}$, $q = \frac{33547}{22201}$.

From which we may conclude that the ſeveral Expectations of *B*, *C*, *D*, *A*, &c. are reſpectively,

Firſt, $4 \times \frac{56}{149} + \frac{45536}{22201}p$; *Secondly*, $4 \times \frac{36}{149} + \frac{38724}{22201}p$.
Thirdly, $4 \times \frac{32}{149} + \frac{27600}{22201}p$; *Fourthly*, $4 \times \frac{25}{149} + \frac{33547}{22201}p$.

The Expectations of the Gameſters being found, it will be neceſſary to find the Riſks of their being fined, or otherwiſe what Sum each of them ought juſtly to give to have their Fines inſured. In order to which, let us form ſo many Schemes as are ſufficient to find the Law of their Proceſs.

And *Firſt*, we may obſerve, that upon the Suppoſition of *B* beating *A* the firſt Game, in conſequence of which *A* is to be fined, *B* and *C* have one Chance each for being fined the ſecond Game, as it thus appears:

$$\begin{array}{c|cc} 1 & BA & BA \\ 2 & \overline{CB} & \overline{BC} \end{array}$$

Secondly, that *C* has one Chance in four for being fined the third Game, *B* one Chance likewiſe, and *D* two; according to the following Scheme.

$$\begin{array}{c|cccc} 1 & BA & BA & BA & BA \\ 2 & \overline{CB} & \overline{CB} & \overline{BC} & \overline{BC} \\ 3 & DC & CD & DB & BD \end{array}$$

Thirdly, that *D* has two Chances in eight for being fined the fourth Game, that *A* has three, and *C* one according to the following Scheme.

$$\begin{array}{c|cccccc} 1 & BA & BA & BA & BA & BA & BA \\ 2 & \overline{CB} & \overline{CB} & \overline{CB} & \overline{CB} & \overline{BC} & \overline{BC} \\ 3 & DC & DC & CD & CD & DB & DB \\ 4 & AD & DA & AC & CA & AD & DA \end{array}$$

N. B. The two Combinations BA, BC, BD, AB, and BA, BC, BD, BA, are omitted in this Scheme as being ſuperfluous; their diſpoſition ſhewing that the Set muſt have been ended in three Games, and conſequently not affecting the Gameſters as to the Probability of their being fined the fourth Game; yet the number of all the Chances muſt be reckoned as being eight; ſince the Probability of any one Circumſtance is but $\frac{1}{8}$.

Theſe Schemes being continued, it will eaſily be perceived that the circumſtances under which the Gameſters find themſelves, in reſpect of their Riſks of being fined, ſtand related to one another in the ſame manner as did their Probabilities of winning; from which conſideration a Table of the Riſks may eaſily be compoſed as follows.

A TABLE *of Riſks.*

		B	*C*	*D*	*A*
I	2	$\frac{1}{2}p$	$\frac{1}{2}p$	-----	-----
II	3	$\frac{1}{4}p$	$\frac{1}{4}p$	$\frac{2}{4}p$	-----
III	4	-----	$\frac{1}{8}p$	$\frac{2}{8}p$	$\frac{3}{8}p$
IV	5	$\frac{3}{16}p$	$\frac{2}{16}p$	$\frac{2}{16}p$	$\frac{3}{16}p$
V	6	$\frac{6}{32}p$	$\frac{5}{32}p$	$\frac{2}{32}p$	$\frac{3}{32}p$
VI	7	$\frac{6}{64}p$	$\frac{8}{64}p$	$\frac{8}{64}p$	$\frac{4}{64}p$
VII	8	$\frac{7}{128}p$	$\frac{8}{128}p$	$\frac{14}{128}p$	$\frac{13}{128}p$
VIII	9	$\frac{17}{256}p$	$\frac{15}{256}p$	$\frac{14}{256}p$	$\frac{22}{256}p$
&c.					

Wherefore ſuppoſing $B^{I} + B^{II} + B^{III}$, &c. $C^{I} + C^{II} + C^{III}$, &c. $D^{I} + D^{II} + D^{III}$, &c. $A^{I} + A^{II} + A^{III}$, &c. to repreſent the ſeveral Probabilities; and ſuppoſing that the ſeveral Sums of theſe Probabilities are reſpectively y, x, z, v, we ſhall have the following Equations $y = \frac{3}{4} + \frac{3}{4}x$; $z = \frac{1}{2} + \frac{1}{2}y + \frac{1}{4}v$; $v = \frac{1}{4} + \frac{1}{2}z + \frac{1}{4}y$; $x = \frac{1}{2}v + \frac{1}{4}z$. Which Equations being ſolved we ſhall have $y = \frac{243}{149}$, $z = \frac{252}{149}$, $v = \frac{224}{149}$, $x = \frac{175}{149}$.

Let now every one of thoſe Fractions be multiplied by p, and the Products $\frac{243}{149}p$, $\frac{252}{149}p$, $\frac{224}{149}p$, $\frac{175}{149}p$ will expreſs the reſpective Riſks of *B*, *C*, *D*, *A*, or the Sums they might juſtly give to have their Fines inſured.

But if from the ſeveral Expectations of the Gameſters there be ſubtracted, *Firſt*, the Sums by them depoſited in the beginning of the Play, and *Secondly*, the Riſks of their Fines, there will remain the clear Gain or Loſs of each. Wherefore

From the Expectations of $B = \frac{224}{149} + \frac{45536}{22201}p.$
Subtracting his own Stake $= 1$
And alſo the Sum of his Riſks $= \frac{243}{149}p.$
There remains his clear Gain $= \frac{75}{149} + \frac{9329}{22201}p.$

From the Expectations of $C = \frac{144}{149} + \frac{38724}{22201}p.$
Subtracting his own Stake $= 1$
And alſo the Sum of his Riſks $= \frac{252}{149}p.$
There remains his clear Gain $= -\frac{5}{149} + \frac{1176}{22201}p.$

From the Expectations of $D = \frac{128}{149} + \frac{37600}{22201}p.$
Subtracting his own Stake $= 1$
And alſo the Sum of his Riſks $= \frac{224}{149}p.$
There remains his clear Gain $= -\frac{21}{149} + \frac{4224}{22201}p.$

From the Expectations of $A = \frac{100}{149} + \frac{33547}{22201}p.$
Subtracting his own Stake $= 1$
And alſo the Sum of his Riſks $= \frac{175}{149}p.$
Laſtly, the Fine due to the Stock by the Loſs of the firſt Game $= p.$
There remains his clear Gain $= -\frac{49}{149} + \frac{14729}{22201}p.$

The foregoing Calculation being made upon the Suppoſition of B beating A in the beginning of the Play, which Suppoſition neither affects C nor D, it follows that the Sum of the Gains between B and A ought to be divided equally; then their ſeveral Gains will ſtand as follows:

$$\text{Gain of}\left\{\begin{array}{lrr} A = & \frac{13}{149} & -\frac{2700}{22201}p \\ B = & \frac{13}{149} & -\frac{2700}{22201}p \\ C = & -\frac{5}{149} & +\frac{1176}{22201}p \\ D = & -\frac{21}{149} & +\frac{4224}{22201}p \end{array}\right.$$

$$\text{Sum of the Gains} = \quad 0 \qquad 0$$

If $\frac{13}{149} - \frac{2700}{22201}p$, which is the Gain of A or B be made $= 0$; then p will be found $= \frac{1937}{2700}$; from which it follows, that if each Man's Stake be to the Fine in the proportion of 2700 to 1937, then A and B are in this caſe neither Winners nor Loſers; but C wins $\frac{1}{225}$ which D loſes.

And in the like manner may be found what the proportion between the Stake and the Fine ought to be, to make C or D play without advantage or diſadvantage; and alſo what this proportion ought to be, to make them play with any advantage or diſadvantage given.

COROLLARY I.

A Spectator S might at firſt, in conſideration of the Sum $4 + 7p$ paid him in hand, undertake to furniſh the four Gameſters with Stakes, and pay all their Fines.

COROLLARY 2.

If the Stock is conſiderably increaſed, and the Gameſters agree either to pay no more Fines, or to give over playing, then

1°. If we ſuppoſe B to have got the laſt Game, by beating out A, and call the Stock *Unity*; the Expectations, or Shares, belonging to B, C, D, A, reſpectively, will be $\frac{56}{149}$, $\frac{36}{149}$, $\frac{32}{149}$, $\frac{25}{149}$.

2°. If B has got 2 Games, by beating D and A ſucceſſively, the Shares of B, C, D, A, are $\frac{87}{149}$, $\frac{28}{149}$, $\frac{18}{149}$, $\frac{16}{149}$. For B has now an equal Chance for the whole Stake, or for the loweſt Chance of the former Caſe: that is, his Expectation is worth

$\frac{1}{2} \times 1 + \frac{25}{149} = \frac{87}{149}$. *C* has an equal Chance for 0, and for $\frac{56}{149}$; that is, his Expectation is $\frac{28}{149}$, and in the ſame way the Numerators of the Expectations of *D* and *A* are found.

COROLLARY 3.

If the proportion of Skill between the Gameſters be given, then their Gain or Loſs may be determined by the Method uſed in this and the preceding Problem.

COROLLARY 4.

If there be never ſo many Gameſters playing on the conditions of this Problem, and the proportion of Skill between them all be ſuppoſed to be equal, then the Probabilities of winning or of being fined may be determined as follows.

Let $\overline{B^{I}}$, $\overline{C^{I}}$, $\overline{D^{I}}$, $\overline{E^{I}}$, $\overline{F^{I}}$, $\overline{A^{I}}$, denote the Probabilities which *B*, *C*, *D*, *E*, *F*, *A* have of winning the Set, or of being fined in any number of Games; and let the Probabilities of winning or of being fined in any number of Games leſs by one than the preceding, be denoted by $\overline{B^{I}}$, $\overline{C^{II}}$, $\overline{D^{II}}$, $\overline{E^{II}}$, $\overline{F^{II}}$, $\overline{A^{II}}$: and ſo on; then I ſay that

$$\begin{aligned}
\overline{B^{I}} &= \tfrac{1}{2}\overline{A^{II}} + \tfrac{1}{4}\overline{A^{III}} + \tfrac{1}{8}\overline{A^{IV}} + \tfrac{1}{16}\overline{A^{V}}\\
\overline{C^{I}} &= \tfrac{1}{2}\overline{B^{I}} + \tfrac{1}{4}\overline{F^{III}} + \tfrac{1}{8}\overline{E^{IV}} + \tfrac{1}{16}\overline{D^{V}}\\
\overline{D^{I}} &= \tfrac{1}{2}\overline{C^{II}} + \tfrac{1}{4}\overline{B^{III}} + \tfrac{1}{8}\overline{F^{IV}} + \tfrac{1}{16}\overline{E^{V}}\\
\overline{E^{I}} &= \tfrac{1}{2}\overline{D^{II}} + \tfrac{1}{4}\overline{C^{III}} + \tfrac{1}{8}\overline{B^{IV}} + \tfrac{1}{16}\overline{F^{V}}\\
\overline{F^{I}} &= \tfrac{1}{2}\overline{E^{II}} + \tfrac{1}{4}\overline{D^{III}} + \tfrac{1}{8}\overline{C^{IV}} + \tfrac{1}{16}\overline{B^{V}}\\
\overline{A^{I}} &= \tfrac{1}{2}\overline{F^{II}} + \tfrac{1}{4}\overline{E^{III}} + \tfrac{1}{8}\overline{D^{V}} + \tfrac{1}{16}\overline{C^{V}}
\end{aligned}$$

Now the Law of theſe relations being viſible, it will be eaſy to extend it to any other number of Gameſters.

COROLLARY 5.

If there be ſeveral Series ſo related to one another, that each Term of one Series may have a certain given proportion to ſome one aſſigned Term in each of the other Series, and that the order of theſe proportions be conſtant and uniform, then will all thoſe Series be exactly ſummable.

REMARK.

As the Application of the Doctrine contained in theſe Solutions and Corollaries may appear difficult when the Gameſters are many, and when it is required to put an end to the play by a fair diſtribution of the money in the *Poule*; which I look upon as the moſt uſeful Queſtion concerning this Game: I ſhall explain this Subject a little more particularly.

1. Let us then Suppoſe any number of Gameſters, $n+1$ (as, in our Scheme, 6) and having written down ſo many Letters as there are Gameſters in the Order they are to ſucceed one another, place under them their reſpective ſmall Letters, to denote the Probabilities which the ſeveral Gameſters have of winning the *Poule*, immediately after their Order of Succeſſion is fixt, and before the play is begun. Where note that the Letter *b* ſignifies ambiguouſly the Expectation of *A* or of *B*: and this Caſe being particular, not to occur again in the ſame *Poule*, may be ſeparated from the others by a line.

			$\left.\begin{matrix}A\\B\end{matrix}\right\}$	C	D	E	F	
	$n-5$	0	b	c	d	e	f	
Number of Games won by *B*			B	C	D	E	F	A
	$n-4$	I	b^{I}	c^{I}	d^{I}	e^{I}	f^{I}	a^{I}
	$n-3$	II	b^{II}	c^{II}	d^{II}	e^{II}	f^{II}	a^{II}
	$n-2$	III	b^{III}	c^{III}	d^{III}	e^{III}	f^{III}	a^{III}
	$n-1$	IV	b^{IV}	c^{IV}	d^{IV}	e^{IV}	f^{IV}	a^{IV}
	n	V	1	0	0	0	0	0

We ſhall always ſuppoſe *B* to be the Winner of the firſt Game; and that *A* takes the loweſt place in the ſecond Row of Capitals. Under theſe repeat $n-1$ Rows of the ſmall Letters which, with the ſmall ſtrokes or dots affixed to them mark the Expectations of the ſeveral Gameſters, when any one Gameſter has got as many Games as is the Number of dots, or that which is marked in Roman Characters to the right of the Row. For it is to be obſerved, that, after the firſt Game, the ſmall Letters thus marked do not, unleſs by accident, ſignify the Expectations of the particular Gameſters at firſt denoted by their Capitals; but the Expectations which belong to the Rank and Column where any Letter ſtands. For Example, b^{III} does not denote the Expectations of him who was ſuppoſed to get the firſt Game, unleſs perhaps he has got two more ſucceſſively; but indefinitely, thoſe of whatever Gameſter has got 3 Games following. And the other Letters of the ſame Row, as c^{III}, d^{III}, e^{III}, ſignify the ſimultaneous Expectations of the three Gameſters that follow him in the Order of playing.

2. This preparation being made, it will be obvious in what manner the Expectations are varied by the Event of every Game; and how they are always reducible to known Numbers.

For if we ſuppoſe *B*, the Gameſter who is in Play, to have got 3 Games, for inſtance, and to want two more of the *Poule*; then his preſent Expectation being b^{III}, if he wins the next Game which he is to play with *C*, the Conſequence will be; 1°. His own Expectations will be changed into b^{IV}; having now got 4 Games. 2°. All the other Expectations in the ſame Row, will likewiſe be transferred to the next inferior (IV.) but marked each by the preceding Letter of the Alphabet: that is, d^{III} becomes c^{IV}, e^{III} becomes d^{IV}, &c. excepting only c^{III}, the Expectation of him who loſt the Game, which is thereby reduced to the loweſt Expectation a^{IV}. And if *B* had already gained $(n-1=)$ 4 Games, and conſequently wanted but one; if he gains this, all the Expectations c^{IV}, d^{IV}, e^{IV}, &c. will vaniſh together, while b^{IV} becomes $=1$, the Exponent of Certainty.

But if *B* loſes his Game with *C*, all the Expectations, of whatever Rank, are transferred to the Rank I, and their Ratios are reſtored as when the firſt Game was won: only the Letters are changed into the next preceding. As b^{III} becomes a^{I}, c^{III} becomes b^{I}, d^{III} becomes c^{I}, and ſo on.

3. Now there being ſuppoſed an equal Chance of B's winning and loſing a Game; any Expectation of his, as when he has got 3 Games, will be thus expreſſed; $b^{III}=\frac{b^{IV}+a'}{2}$; in which, ſubſtituting for b^{IV} its equal in our Example $\frac{1+a'}{2}$, we ſhall have $b^{III}=\frac{1+3\times a'}{4}$. The ſame way, $b^{II}=\frac{b^{III}+a'}{2}=\frac{1+7a'}{8}$; and $b'=\frac{1+15a'}{16}$. *In general*; when the number of Games that *B* wants of gaining the *Poule* is *m*, then ſhall $\frac{1+\overline{2^m-1}\times a'}{2^m}$ be the value of his Expectations.

4. The other Expectations are collected nearly in the ſame manner. As $c^{III}=\frac{a^{IIII}+b^{I}}{2}$, in which ſubſtituting for a^{IIII} its equal (in our example) $\frac{0+f'}{2}$, we have $c^{III}=\frac{f'+2b'}{4}=\frac{1}{2}b^{I}+\frac{1}{4}f^{I}$. The ſame way, $c^{II}=\frac{1}{2}b^{I}+\frac{1}{4}f^{I}+\frac{1}{8}e^{I}$: and $c^{I}=\frac{1}{2}b^{I}+\frac{1}{4}f^{I}+\frac{1}{8}e^{I}+\frac{1}{16}d^{I}$; the number of Terms added to the Games won being always $=n$, and the Letter a^{I} always omitted.

From all which it appears, that the Expectation of a Gameſter, in any State of the Play, is expreſſed by the Expectations a', b', c', &c. after one Game is won: and that theſe, therefore, are firſt to be computed.

5. In order to which, I ſay, that the Letters b, c, d, e, &c. expreſſing, as above, the Chances of the Gameſters, B, C, D, E, &c. immediately after their Order of playing is fixt by lot, or otherwiſe; theſe Chances are in the geometrical Progreſſion of $1+2^n$ to 2^n.

For either of the Gameſters (as A) who play the firſt Game, has 1 out of 2^n Chances of beating all his Adverſaries in one Round. And therefore he may, in conſideration of the Sum $\frac{1}{2^n}\times\overline{b+c+d+e}$ give up his expectations ariſing from the Probability of that Event, and take the loweſt place with the Expectation e; the Gameſter C ſucceeding to his place, D to that of C; and ſo on. But B having, on the ſcore of Priority, the ſame demand upon A, as A has upon B; that is, neither having any demand upon the other, the Term $\frac{1}{2^n}\times b$ is to be cancelled; and the Value of A's place, with reſpect to the other Gameſters, reduced to $\frac{1}{2^n}\times c+\frac{1}{2^n}\times d+$ &c. And now each of the Gameſters C, D, E, &c. being raiſed to the next higher Expectation b, c, d, &c. for which he has paid $\frac{1}{2^n}$ of his former Expectation; it follows that $b=\overline{1+\frac{1}{2^n}}\times c$, $c=\overline{1+\frac{1}{2^n}}\times d$, &c. and that, before the play is begun, every Expectation is to the next below it as $1+\frac{1}{2^n}$ to 1, or as $1+2^n$ to 2^n. Which coincides with Theor. I. of Mr. *Nicolas Bernoulli* in Phil. Tranſ. N. 341.

Thus if the Gameſters are 3, (A) B, C; their firſt Expectations are (5) 5, 4, with the common Denominator 14. If they are 4, (A) B, C, D, their Expectations are (81) 81, 72, 64, with the Denominator 298. If there are 5 Gameſters, their Expectations are (17^3) 17^3, $17^2\times16$, 17×16^2, 16^3, with their Sum for a Denominator; that is, (4913), 4913, 4624, 4352, 4096, with the Denominator 22898. And the like for any number of Gameſters.

6. It is plain likewiſe that the Expectations of all the Gameſters, excepting A and B, remain the ſame after one Game is plaid, as they were at firſt; $c'=c$, $d'=d$, $e'=e$, &c. becauſe the conteſt

in the firſt Game concerns A and B alone; its Event making no alteration in the Expectations of the others: but only raiſing B's firſt expectation, which was b, to the Value b', and diminiſhing the equal Expectation of A by the ſame quantity: ſo that $a' + b' = 2b$.

And therefore, to find all the Expectations after the firſt Game is played, we have now only to compute the firſt and laſt of that Rank, b' and a'.

But it was found already that if m repreſents the number of Games that the laſt Winner B wants to gain the *Poule*, his Expectations in that Circumſtance will be equal to $\frac{1+2^m-1\times a'}{2^m}$. From which, putting $m = n - 1$, which is the Caſe when B has got one Game, and the Expectation b'; and ſubſtituting for b' its equal $2b - a'$, we ſhall get $a' = \frac{2^n b - 1}{2^n - 1}$:

As when there are 3 Gameſters, $n = 2$, $b = \frac{5}{14}$ and $a' = \frac{\frac{20}{14}-1}{3} = \frac{6}{42} = \frac{1}{7}$. And $b' = 2b - a' = \frac{10}{14} - \frac{2}{14} = \frac{4}{7}$.

If there are 4 Gameſters, $n = 3$, $b = \frac{81}{298}$; and therefore $a' = \overline{8 \times \frac{81}{298} - 1} \times \frac{1}{7} = \frac{350}{298} \times \frac{1}{7} = \frac{50}{298} = \frac{25}{149}$. And $b' = \frac{81}{149} - \frac{25}{149} = \frac{56}{149}$.

If there are 5 Gameſters, $n = 4$, $b = \frac{4913}{22898}$; whence $a' = \overline{16 \times \frac{4913}{22898} - 1} \times \frac{1}{15} = \frac{55710}{22898} \times \frac{1}{15} = \frac{3714}{22898} = \frac{1857}{11449}$. And $b' = 2b - a' = \frac{4913}{11449} - \frac{1857}{11449} = \frac{3056}{11449}$. So that the Expectations of the Gameſters, B having got one Game, will ſtand thus:

B	C	D	E	A
b'	c'	d'	e'.	a'.

3056, 2312, 2176, 2048, 1857; theſe numbers expreſſing the Ratios of the Expectations; and with the Denominator 11449 ſubſcribed, their abſolute quantity; or the Shares of the whole Stake due to each Gameſter, if they were to give over playing.

7. And thus the Probabilities which the ſeveral Gameſters have of gaining the *Poule* may in all Caſes be computed, and diſpoſed into Tables. But the 6 following, will, 'tis thought, be more than ſufficient for any Caſe that happens in play.

TABLE I. For a *Poule* of *Three*.

Games won	*A* } *B* }	*C*	
0	5	4	Denom. 5+5+4=14.

	B	*C*	*A*	
I.	4	2	1	Denom. 7.

Tab. II. For a *Poule* of *Four*.

	A } *B* }	*C*	*D*	
0	81	72	64	Denom. 298.

	B	*C*	*D*	*A*	Denom.
I.	56	36	32	25	149.
II.	87	28	18	16	

Tab. III. For a *Poule* of *Five*.

	A } *B* }	*C*	*D*	*E*	Denom.
0	4913	4624	4352	4096	22898.

	B	*C*	*D*	*E*	*A*	
I.	3056	2312	2176	2048	1857	Denom. 11449.
II.	4255	2040	1920	1666	1568	
III.	6653	1528	1156	1088	1024	

Tab. IV. For a *Poule* of *Six*.

	A } *B* }	*C*	*D*	*E*	*F*	Denom.
0	1185921	1149984	1115136	1081344	1048576	6766882.

	B	*C*	*D*	*E*	*F*	*A*	
I.	682976	574992	557568	540672	524288	502945	Denom. 3383441.
II.	863007	540144	523776	507904	481602	467008	
III.	1223069	472560	458240	422532	409728	397312	
IV.	1943193	341488	287496	278784	270336	262144	

One Example will ſhew the Uſe of the Tables: Suppoſe 5 Gameſters engaged in a *Poule*, with this condition, that if it is not ended when a certain number of Games are played, they ſhall give over, and divide the Money in proportion to the Chances they ſhall then have of winning the *Poule*. That number of Games being played, ſuppoſe the *Poule* riſen to 30 Guineas, and that a Gameſter (*B*) has got two Games: *Qu.* how the 30 Guineas are to be ſhared?

Divide $31\frac{1}{2}\,l.$ into Shares proportional to the numbers 4255, 2040, &c. (in Tab. III.) which ſtand in the Row of *Games won* II. and thoſe Shares will be as follow: $31\frac{1}{2}\,l. \times \frac{4255}{11449}$ = the Share of

And the ſame way thoſe of *C*, &c. will be - - -

	l.	*s.*	*d.*
B	11	14	2
C	5	12	3
D	5	5	8
E	4	11	8
A	4	6	3
L.	31	10	0

Note, the pricked Line which is drawn in each of the Tables ſeparates the Chances of the Gameſters who are *neceſſarily* to come into the play before the *Poule* is won, from the Chances of thoſe who *may poſſibly* not come in again; which lie below that line. And, ſetting aſide the Column *B*, all the Chances in any Row above the line are in the continued Ratio of $\overline{1+2^n}$ to 2^n. As in Tab. III. $d'' = \frac{16}{17} \times c''$, or $1920 = \overline{1 - \frac{1}{17}} \times 2040$.

The ſame is true of the Terms of any Row that lie both below the line. But if one lies above and the other below it, their Relation is different, and is to be found by *Art.* 3. of this Remark.

It remains to compute the Profit and Loſs upon the Fines p: as follows.

1. The preſent Expectations of a Gameſter who is *entering*, or to enter, into play, that he ſhall be the Winner, are made up of his ſeveral preſent Expectations, upon the Events of his coming in *once*, *twice*, *thrice*, &c. as is manifeſt. And as, immediately after the Order of playing is fixt, it was ſhewn that thoſe *total* Expectations are in the geometrical Progreſſion of $1+2^n$ to 2^n, the number of Gameſters being $n+1$; ſo, in any other State of the *Poule*, their Ratio is always given.

But every time that a Gameſter enters, his Chance of *winning* in that *Turn*, is to his Chance of paying a *Fine*, as 1 to $2^n - 1$: and therefore, *componendo*, the Sum of a Gameſter's ſeveral Expectations of *winning*, is to the Sum of his ſeveral Riſks of paying a *Fine*, in the ſame Ratio; the whole Stake, and alſo each *Fine* p, being put $= 1$. And the whole *Riſks* of the ſeveral Gameſters are in the ſame Ratios as their *Expectations*.

Thus in the Caſe of *Three* Gameſters, whoſe Expectations are $\frac{5}{14}$, $\frac{5}{14}$, $\frac{4}{14}$, their Chances of paying the Fine p will be the ſame Fractions multiplied into 3 $(= 2^n - 1)$; that is, they will be $\frac{15}{14}$, $\frac{15}{14}$, $\frac{12}{14}$.

And the firſt Expectations of *Four* Gameſters being 81, 81, 72, 64, to the Denominator 298; their Chances of being *Fined* will be the ſame Numerators multiplied into 7 $(= 2^n - 1)$, that is, $\frac{567}{298}$, $\frac{567}{298}$, $\frac{504}{298}$, $\frac{448}{298}$; reſpectively.

Hence again it appears, that the Total of the *Fines*, or the Sum for which they may be furniſhed throughout the *Poule*, is $\overline{2^n - 1} \times p$. For the Sum of the Expectations upon the Stake 1, is 1; and theſe are to the Number of *Fines* as 1 to $2^n - 1$.

2. Suppoſe now that one of the firſt Players of *Three*, as *A*, is beat out, and his *Fine* paid, as muſt always neceſſarily happen; and thence, the Expectations of getting the *Poule* reduced to $\overset{C}{\frac{4}{7}}\ \overset{B}{\frac{2}{7}}\ \overset{A}{\frac{1}{7}}$: then the Riſks of *C* and *A* will be $\frac{6}{7}$, $\frac{3}{7}$, reſpectively: whoſe Sum $\frac{9}{7}$ taken from 2 $(= 2^n - 2)$ leaves $\frac{5}{7}$ for the *Fines* of *B*.

In like manner, the Expectations of *Four* Gameſters, after one Game is won, being 56, 36, 32, 25, with the Denominator 149; the Numerators of the Riſks of the Three laſt Gameſters *C*, *D*, *A*, will be 36, 32, 25, multiplied by 7 $(= 2^n - 1)$ to the ſame Denominator; and their Sum taken from the *Fines* to be paid after one Game is won, which are $6 = 2^n - 2$, leaves for the Riſks of *B*, $\frac{243}{149}$: thoſe of *C*, *D*, *A*, being $\frac{252}{149}$, $\frac{224}{149}$, $\frac{175}{149}$, reſpectively.

3. If *B* has got more than one Game, the Sums for which a Spectator *R* may furniſh all the ſubſequent *Fines*, will be found as follows.

Let the Number of *Fines* which *R* riſks to pay, when *B* has got 1, 2, 3, 4, &c. Games, be x, y, z, v, &c. reſpectively; then $\frac{\overline{x+1}+\overline{y+1}}{2}=x$; or $y=x-2$, $\frac{\overline{x+1}+\overline{z+1}}{2}=y=x-2$; or $z=x-\overline{2^2+2}$. And the ſame way $v=x-\overline{2^3+2^2+2}$, &c.; in an obvious Progreſſion.

Becauſe when *B* has got 1 Game, there is an equal Chance of his winning or loſing the next; in the former Caſe, *R* pays the *Fine* $1\times p$ for *C*, and comes to have the Riſk y; but if *C* wins, *R* pays $1\times p$ for *B*, and his Riſk x is the ſame as before: and ſo of the reſt. So that the number of pieces p for which *R* may engage to furniſh the ſubſequent *Fines*, when *B* has got 2, 3, 4, &c. Games, is had *by the continual Subtraction of* 2 *and its Powers from* 2^n-2. As in a *Poule* of four, when *B* has got 2 Games, the Sum of the Riſks is $6-2=4$. In a *Poule* of five, $x=2^n-2=14$, $y=12$, $z=8$, $v=0$.

And from theſe numbers ſubtracting the Riſks of the other Gameſters *C*, *D*, *E*, &c. found as above, there will remain the Riſks of *B* the Gameſter who continues in play.

4. The Expectations of the ſeveral Gameſters upon the *Fines* may likewiſe be determined by an obvious, but more troubleſome, Operation.

Under the Capitals, B C D E A, write their ſmall Letters thus:

$$\begin{array}{llllll} \text{I.} & b' & c' & d' & e' & a' \\ \text{II.} & b'' & c'' & d'' & e'' & a'' \\ \text{III.} & b''' & c''' & d''' & e''' & a''' \\ \text{IV.} & b^{\text{iv}} & 0 & 0 & 0 & 0 \end{array}$$

Signifying, reſpectively, the Number of Fines which a Gameſter, winning the *Poule*, may expect to find in it, *B* having already got ſo many Games as the *Dots* affixed to the Letter: and to theſe Letters prefix their fractional Coefficients taken from the *Tables of Probabilities*. Then, by the law of the Game, there will be formed a Series of Equations determining the Expectations ſought.

As in the Caſe of 3 Gameſters, write,

$$\left.\begin{array}{llll} & B & C & A \\ \text{I.} & \frac{4}{7}b' & \frac{2}{7}c' & \frac{1}{7}a' \\ \text{II.} & 1\times b''=2 & 0 & 0 \end{array}\right\}\text{and the Equations}\left\{\begin{array}{l} 1^\circ.\ \frac{4}{7}b'=\frac{1}{2}\times\overline{2+\frac{1}{7}\times\overline{a'+1}} \\ 2^\circ.\ \frac{2}{7}c'=\frac{1}{2}\times\overline{\frac{4}{7}\times\overline{b'+1}+0} \\ 3^\circ.\ \frac{1}{7}a'=\frac{1}{2}\times\overline{\frac{2}{7}\times\overline{c'+1}+0} \end{array}\right.$$

Which being reduced give $B' \left(= \frac{4}{7} b'\right) = \frac{68}{49}$, $C' \left(= \frac{2}{7} c'\right) = \frac{48}{49}$, $A' \left(= \frac{1}{7} a'\right) = \frac{31}{49}$. From which ſubtracting their reſpective Riſks; for *B*, $\frac{35}{49}$; for *C*, $\frac{42}{49}$; and for *A*, $\frac{70}{49}$ ($= \frac{3}{7} + 1$, his *Fines*, and the *Fine* already paid in), remain the Gains $+\frac{33}{49}$, $+\frac{6}{49}$, $-\frac{39}{49}$, multiplied into p.

If it is a *Poule* of four, the Expectations on the *Fines* will ſtand thus:

$$\begin{array}{lcccc} & B & C & D & A \\ \text{I.} & \frac{56}{149} b' & \frac{36}{149} c' & \frac{32}{149} d' & \frac{25}{149} a' \\ \text{II.} & \frac{87}{149} b'' & \frac{28}{149} c'' & \frac{18}{149} d'' & \frac{16}{149} a'' \\ \text{III.} & 1 \times b''' = 3 & 0 & 0 & 0 \end{array}$$

and ſetting aſide the common Denominator 149, the Equations will be;

$$1°.\ 56b' = \tfrac{1}{2} \times \overline{25.\overline{a'+1}+87b''} \qquad 5°.\ 28c'' = \tfrac{1}{2} \times 56\,\overline{b'+2}+0 \text{ i. e. } c''=b'+2$$

$$2°.\ 36c' = \tfrac{1}{2} \times \overline{56.\overline{b'+1}+16a''} \qquad 6°.\ {}''d = c'+2$$

$$3°.\ 32d' = \tfrac{1}{2} \times \overline{36.\overline{c'+1}+28c''} \qquad 7°.\ a'' = d'+2$$

$$4°.\ 25a' = \tfrac{1}{2} \times \overline{32.\overline{d'+1}+18\,{}''d} \qquad 8°.\ \frac{87}{149} b'' = \tfrac{1}{2} \times \overline{3+\frac{25}{149}\overline{a'+1}}.$$

Whence will be found

$$\left.\begin{array}{ll} b' = \dfrac{813\frac{1}{7}}{149}; \text{ and } \left(\dfrac{56}{149} b' =\right) & B' = \dfrac{45536}{22201} \\ c' = \dfrac{1075\frac{2}{3}}{149}; \text{ and} & C' = \dfrac{38724}{22201} \\ d' = \dfrac{1175}{149}; \text{ and} & D' = \dfrac{37600}{22201} \\ a' = \dfrac{1341\frac{22}{25}}{149}; \text{ and} & A' = \dfrac{33547}{22201} \end{array}\right\} \quad \left\{\begin{array}{l} b'' = \dfrac{618\frac{34}{87}}{149}; \text{ and } B'' = \dfrac{53800}{22201} \\ c'' = \dfrac{1111\frac{1}{7}}{149}; \text{ and } C'' = \dfrac{31112}{22201} \\ {}''d = \dfrac{1373\frac{2}{3}}{149}; \text{ and } D'' = \dfrac{24726}{22201} \\ a'' = \dfrac{1473}{149}; \text{ and } A'' = \dfrac{23568}{22201}. \end{array}\right.$$

And the like Computations may be made for the ſuperior *Poules*; the Compoſition of the Equations to be reduced being regular and obvious.

PRO-

PROBLEM XLVI.

Of HAZARD.

To find at Hazard the Advantage of the Setter *upon all Suppoſitions of Main and Chance.*

SOLUTION.

Let the whole Money played for be conſidered as a common Stake, upon which both the Caſter and the Setter have their ſeveral Expectations; then let thoſe Expectations be determined in the following manner.

Firſt, Let it be ſuppoſed that the Main is VII: then if the Chance of the Caſter be VI or VIII, it is plain that the Setter having 6 Chances to win, and 5 to loſe, his Expectation will be $\frac{6}{11}$ of the Stake: but there being 10 Chances out of 36 for the Chance to be VI, or VIII, it follows, that the Expectation of the Setter reſulting from the Probability of the Chance being VI or VIII, will be $\frac{10}{30}$ multiplied by $\frac{6}{11}$ or $\frac{60}{11}$ divided by 36.

Secondly, If the Main being VII, the Chance ſhould happen to be V or IX, the Expectation of the Setter would be $\frac{24}{5}$ divided by 36.

Thirdly, If the Main being VII, the Chance ſhould happen to be IV or X, it follows that the Expectation of the Setter would be 4 divided by 36.

Fourthly, If the Main being VII, the Caſter ſhould happen to throw II, III, or XII, then the Setter would neceſſarily win, by the Law of the Game; but there being 4 Chances in 36 for throwing II, III, or XII, it follows that before the Chance of the Caſter is thrown, the Expectation of the Setter reſulting from the Probability of the Caſter's Chance being II, III, or XII, will be 4 divided by 36.

Laſtly, If the Main being VII, the Caſter ſhould happen to throw VII, or XI, the Setter loſes his Expectation.

From the Solution of the foregoing particular Caſes it follows, that the Main being VII, the Expectation of the Setter will be expreſſed by the following Quantities, *viz.* $\frac{\frac{60}{11}+\frac{24}{5}+\frac{4}{1}+\frac{4}{1}}{36}$ which may be reduced to $\frac{251}{495}$; now this fraction being ſubtracted from Unity, to which the whole Stake is ſuppoſed equal, there will remain the Expectation of the Caſter, *viz.* $\frac{244}{495}$.

But the Probabilities of winning being always proportional to the Expectations, on Suppoſition of the Stake being fixt, it follows that the Probabilities of winning for the Setter and Caſter are reſpectively proportional to the two numbers 251 and 244, which properly denote the Odds of winning.

Now if we ſuppoſe each Stake to be 1, or the whole Stake to be 2, the Gain of the Setter will be expreſſed by the fraction $\frac{7}{495}$, it being the difference of the numbers expreſſing the Odds, divided by their Sum, which ſuppoſing each Stake to be a Guinea of 21 Shillings will be about $3^{d.} - 2\frac{1}{4}f$.

By the ſame Method of Proceſs, it will be found that the Main being VI or VIII, the Gain of the Setter will be $\frac{167}{7128}$ which is about $5^{d.} - 3\frac{1}{2}f$ in a Guinea.

It will be alſo found that the Main being V or IX, the Gain of the Setter will be $\frac{43}{2835}$, which is about $3^{d.} - 3\frac{1}{3}f$ in a Guinea.

COROLLARY 1.

If each particular Gain made by the Setter, in the Caſe of any Main, be reſpectively multiplied by the number of Chances which there are for that Main to come up, and the Sum of the Products be divided by the number of all thoſe Chances, the Quotient will expreſs the Gain of the Setter before a Main is thrown: from whence it follows that the Gain of the Setter, if he be reſolved to ſet upon the firſt Main that may happen to be thrown, is to be eſtimated by $\frac{42}{495}+\frac{1670}{7128}+\frac{344}{2835}$, the whole to be divided by 24, which being reduced will be $\frac{37}{2010}$, or about $4^{d.} - 2\frac{1}{2}f$ in a Guinea.

COROLLARY 2.

The Probability of no Main, is to the Probability of a Main as 109 + 2 to 109 − 2, or as 111 to 107.

COROLLARY 3.

If it be agreed between the Caſter and Setter, that the Main ſhall always be VII, and it be farther agreed, that the next Chance happening to be Ames-ace, the Caſter ſhall loſe but half his Stake, then the Caſter's Loſs is only $\frac{1}{3960}$ of his Stake, that is about $\frac{1}{4}f$ in a Guinea.

COROLLARY 4.

The Main being VI or VIII, and the Caſter has $\frac{3}{4}$ of his money returned in caſe he throws Ames-ace, what is his Loſs? And if the Main being V or IX, and he has $\frac{1}{2}$ of his Money returned in caſe he throws Ames-ace, what is his Loſs? In anſwer to the firſt, the Gain of the Setter or Loſs of the Caſter is $\frac{1}{385\frac{11}{37}}$.

In anſwer to the ſecond the Loſs of the Caſter would be but $\frac{1}{782.\frac{2}{29}}$.

COROLLARY 5.

If it be made a ſtanding Rule, that whatever the Main may happen to be, if the Caſter throws Ames-ace immediately after the Main, or in other words, if the Chance be Ames-ace, the Caſter ſhall only loſe $\frac{1}{3}$ of his own Stake, then the Play will be brought ſo near an Equality, that it will hardly be diſtinguiſhable from it; the Gain of the Caſter being upon the whole but $\frac{1}{6048}$ of his own Stake, or $\frac{1}{6}$ of a farthing in a Guinea.

The Demonſtration of this is eaſily deduced from what we have ſaid before *viz.* that the Loſs of the Caſter is $\frac{37}{2016}$; now let us conſider what part of his own Stake ſhould be returned him in caſe he throws Ames-ace next after the Main; Let z be that part, but the Probability of throwing Ames-ace next after the Main is $\frac{1}{36}$, therefore, the real Value of what is returned him is $\frac{1}{36}z$, and ſince the Play is ſuppoſed to be reduced to an Equality, then what is returned him muſt equal his Loſs; for which reaſon, we have the Equation $\frac{z}{36} = \frac{37}{2016}$, or $z = \frac{37}{56}$ which being very near $\frac{2}{3}$, it follows that $\frac{2}{3}$ of his own Stake ought to be returned him.

Or thus; if the Caſter has returned him $\frac{37}{56}$ when that happens, he loſes nothing; but there being but 1 Chance in 36 for that Caſe to happen; the real Value of what is returned is but $\frac{37}{56\times36}$; and in the ſame manner if $\frac{2}{3}$ is returned, the real Value is $\frac{2}{3\times36}$: and ſo, the Difference $\frac{2}{3\times36}-\frac{37}{56\times36}=\frac{1}{6048}$ is the Gain of the Caſter.

PROBLEM XLVII.

To find at Hazard the Gain of the Box for any number of Games diviſible by 3.

SOLUTION.

Let *a* and *b* reſpectively repreſent the Chances for winning a *Main* or for loſing it, which is uſually called a *Main* and no *Main*; then,

1°, It is very viſible that when the four laſt Mains are *baaa*, otherwiſe that when a Main has been loſt, if the three following Mains are won ſucceſſively, then the Box muſt be paid.

2°, That the laſt 7 Mains being *baaaaaa*, there is alſo a Box to be paid.

3°, That the laſt 10 Mains being *baaaaaaaaa*, the Box is to be paid, and ſo on.

Now the Probability of the 4 laſt Mains being *baaa* is $\frac{ba^3}{\overline{a+b}|^4}$, and conſequently, if the number of Mains thrown from the beginning is repreſented by *n*, the Gain of the Box upon this account will be $\frac{\overline{n-3}\times ba^3}{\overline{a+b}|^4}$.

But to obviate a difficulty which may perhaps ariſe concerning the foregoing Expreſſion which one would naturally think muſt be $\frac{nba^3}{\overline{a+b}|^4}$, it muſt be remembered that the Termination *baaa* belongs to 4 Games at leaſt, and that therefore the three firſt Games are to be excluded from this Caſe, tho' they ſhall be taken notice of afterwards.

Again the Probability of the 7 laſt Mains terminating thus *baaaaaa*, will be $\frac{ba^6}{\overline{a+b}|^{7}}$, but this Caſe does not belong to the 6 firſt Mains, therefore the Gain of the Box upon this account will be $\frac{\overline{n-6}\times ba^6}{\overline{a+b}|^{7}}$; and ſo on.

And therefore the firſt part of the Expectation of the Box is expreſſed by the Series

$$\frac{\overline{n-3}\times ba^3}{\overline{a+b}|^{4}}+\frac{\overline{n-6}\times ba^6}{\overline{a+b}|^{7}}+\frac{\overline{n-9}\times ba^9}{\overline{a+b}|^{10}}+\frac{\overline{n-12}\times ba^{12}}{\overline{a+b}|^{13}},\ \&c.$$

of which the number of Terms is $\frac{n-3}{3}$.

The ſecond part of the Expectation of the Box ariſes from all the Mains being won ſucceſſively without any interruption of a no Main, and this belongs particularly to the three firſt Mains, as well as to all thoſe which are diviſible by 3, and therefore the ſecond part of the Expectation of the Box will be expreſſed by the Series $\frac{a^3}{\overline{a+b}|^{3}}+\frac{a^6}{\overline{a+b}|^{5}}+\frac{a^9}{\overline{a+b}|^{9}}+\frac{a^{12}}{\overline{a+b}|^{12}}$, &c. of which the number of Terms is $\frac{n}{3}$.

Thoſe who will think it worth their while to ſum up theſe Series, may without much difficulty do it, if they pleaſe to conſult my *Miſcellanea*, wherein ſuch ſorts of Series, and others more compound, are largely treated of.

In the mean time, I ſhall here give the Reſult of what they may ſee there demonſtrated.

If the firſt Series be diſtinguiſhed into two others, the firſt poſitive, the other negative, we ſhall now have three Series, the Sums of which will be, ſuppoſing $\frac{a}{a+b}=r$.

1°, $\frac{nb}{a+b}\times\frac{\overline{r^3-n^3}}{1-r^3}$

2°, $-\frac{3b}{a+b}\times\frac{\overline{r^3-\frac{1}{3}nr^{n}+\frac{1}{3}\times\overline{n-1}\times r^{n+3}}}{\overline{1-r^3}|^{2}}$

3°, $\frac{r^3-r^{n+3}}{1-r^3}$

the ſum of all which will be reduced to the Expreſſion $\frac{n}{14}-\frac{5}{49}+\frac{5}{49\times 2^{n}}$, when *a* and *b* are in a Ratio of Equality.

COROLLARY I.

If *n* be an infinite number, the Gain of the Box will be univerſally expreſſed by $\frac{nb}{a+b}\times\frac{a^3}{\overline{a+b}|^{3}-a^3}$; but when *a* and *b* are in a Ratio of Equality by $\frac{n}{14}$.

COROLLARY 2.

The Gain of the Box being ſuch as has been determined for an infinite number of Mains, it follows that, one with another, the Gain of the Box for one ſingle Main ought to be eſtimated by $\frac{b}{a+b}\times\frac{a^3}{\overline{a+b}^{\,3}-a^3}$, or $\frac{1}{14}$ if a and b are equal.

COROLLARY 3.

And conſequently, it follows that in ſo many Mains as are expreſſed by $\frac{\overline{a+b}\times\overline{a+b}^{\,3}-a^3}{a^3b}$, or in 14 Mains if a and b are equal, the Expectation of the Box is 1, calling 1 whatever is ſtipulated to belong to the Box, which uſually is 1 Half-Guinea.

COROLLARY 4.

Now ſuppoſing that a and b are reſpectively as 107 to 111, a Box is payed one with another in about 14.7 Mains.

After I had ſolved the foregoing Problem, which is about 12 years ago, I ſpoke of my Solution to Mr. *Henry Stuart Stevens*, but without communicating to him the manner of it: As he is a Gentleman who, beſides other uncommon Qualifications, has a particular Sagacity in reducing intricate Queſtions to ſimple ones, he brought me, a few days after, his Inveſtigation of the Concluſion ſet down in my third Corollary; and as I have had occaſion to cite him before, in another Work, ſo I here renew with pleaſure the Expreſſion of the Eſteem which I have for his extraordinary Talents: Now his Inveſtigation was as follows.

Let a and b reſpectively repreſent the number of Chances for a Main and no Main; Let alſo 1 be the Sum which the Box muſt receive upon Suppoſition of three Mains being won ſucceſſively; now the Probability of winning a Main is $\frac{a}{a+b}$, and the Probability of winning three Mains is $\frac{a^3}{\overline{a+b}^{\,3}}$, and therefore the Box-keeper might without advantage or diſadvantage to himſelf receive from the Caſter at a certainty, the Sum $\frac{a^3}{\overline{a+b}^{\,3}}\times 1$, which would be an Equivalent for the uncertain ſum 1, payable after three Mains.

Let it therefore be agreed between them, that the Caſter ſhall pay but the Sum $\frac{a^3}{\overline{a+b}^3} \times 1$ for his three Mains; now let us ſee what conſideration the Box-keeper gives to the Caſter in return of that Sum. 1°, he allows him one Main ſure, 2°, he allows him a ſecond Main conditionally, which is provided he wins the firſt, of which the Probability being $\frac{a}{a+b}$, it follows that the Box allows him only, if one may ſay ſo, the portion $\frac{a}{a+b}$ of a ſecond Main, and for the ſame reaſon the portion $\frac{aa}{\overline{a+b}^2}$ of a third Main, and therefore the Box allows in all to the Caſter $1 + \frac{a}{a+b} + \frac{aa}{\overline{a+b}^2}$ Mains, or $\frac{3aa+3ab+bb}{\overline{a+b}^2}$; and therefore if for the Sum received $\frac{a^3}{\overline{a-b}^3} \times 1$, there be the allowance of $\frac{3aa+3ab+bb}{\overline{a+b}^2}$ Mains, how many are allowed for the Sum 1? and the Term required will be $\frac{\overline{3aa+3ab+bb} \times \overline{a+b}}{a^3}$, or $\frac{\overline{a+b}^4}{ab^3} - \frac{a+b}{b}$: and therefore in ſo many Mains as are denoted by the foregoing Expreſſion, the Box gets the Sum 1; which Expreſſion is reduced to 14 if a and b are equal.

PROBLEM XLVIII.

Of RAFFLING.

If any number of Gameſters A, B, C, D, *&c. play at* Raffles, *what is the Probability that the firſt of them having thrown his Chance, and before the other Chances are thrown, wins the Money of the Play?*

SOLUTION.

In order to ſolve this Problem, it is neceſſary to have a Table ready compoſed of all the Chances which there are in three *Raffles*, which Table is the following.

A TABLE *of all the Chances which are in three Raffles.*

Points.			Chances to win or loſe.	Chances to win or loſe.	Equality of Chance.
LIV		IX	884735	0	1
LIII		X	884726	1	9
LII		XI	884681	10	45
LI		XII	884534	55	147
L		XIII	884165	202	369
XLIX		XIV	883400	571	765
XLVIII		XV	881954	1336	1446
XLVII		XVI	879470	2782	2484
XLVI		XVII	875501	5266	3969
XLV		XVIII	869632	9235	5869
XLIV		XIX	861199	15104	8433
XLIII	or	XX	849706	23537	11493
XLII		XXI	834679	35030	15027
XLI		XXII	815392	50057	19287
XL		XXIII	791506	69344	23886
XXXIX		XXIV	762838	93230	28668
XXXVIII		XXV	728971	121898	33867
XXXVII		XXVI	690100	155765	38871
XXXVI		XXVII	646929	194636	43171
XXXV		XXVIII	599472	237807	47457
XXXIV		XXIX	548865	285264	50607
XXXIII		XXX	496314	335871	52551
XXXII		XXXI	442368	388422	53946

The Sum of all the numbers expreſſing the Equality of Chance being 442368, if that Sum be doubled it will make 884736, which is equal to the Cube of 96.

The firſt Column contains any number of Points which *A* may be ſuppoſed to have thrown in three Raffles.

The ſecond Column contains the number of Chances which *A* has for winning, if his Points be above XXXI, or the number of Chances he has for loſing, if his Points be either XXXI or below it.

The third Column contains the number of Chances which *A* has for loſing, if his Points be above XXXI, or for winning, if they be either XXXI or below it.

The fourth Column, which is the principal, and out of which the other two are formed, contains the number of Chances whereby any number of Points from ix to liv can be produced in three Raffles; and conſequently contains the number of Chances which any of the Gameſters *B*, *C*, *D*, &c. may have for coming to an equality of Chance with *A*.

The Conſtruction of the fourth Column depends chiefly on the number of Chances which there are for producing one ſingle Raffle, whereof xviii or iii have 1 Chance
xvii or iv have 3 Chances
xvi or v have 6 Chances
xv or vi have 4 Chances
xiv or vii have 9 Chances
xiii or viii have 9 Chances
xii or ix have 7 Chances
xi or x have 9 Chances

Which number of Chances being duly combined, will afford all the Chances of three Raffles.

But it will be convenient to illuſtrate this by one Inſtance; let it therefore be required to find the number of Chances for producing xii Points in three Raffles.

1°, It may plainly be perceived that thoſe Points may be produced by the following ſingle Raffles iii, iii, vi, or iii, iv, v, or iv, iv, iv; then conſidering the firſt Caſe, and knowing from the Table of ſingle Raffles, that the Raffles iii, iii, vi, have reſpectively 1, 1, 4 Chances to come up, it follows from the Doctrine of Combinations that thoſe three numbers ought to be multiplied together, which in the preſent Caſe makes the product to be barely 4, but as the diſpoſition, iii, iii, vi, may be varied twice; *viz.* by iii, vi, iii, and vi, iii, iii, which will make in all three diſpoſitions, it follows that the number 4, which expreſſes the Chances of one diſpoſition, ought to be multiplied by 3, which being done, the product 12 muſt be ſet apart.

2°, The Diſpoſition iii, iv, v, has for its Chances the product of the numbers 1, 3, 6, which makes 18; but this being capable of 6 permutations, the number 18 ought to be multiplied by 6, which being done, the product 108 muſt likewiſe be ſet apart.

3°, The Diſpoſition iv, iv, iv has for its Chances the product of 3, 3, 3, which makes 27; but this not being capable of any variation, we barely write 27, which muſt be ſet apart.

4°, Adding together thoſe numbers that were ſeverally ſet apart, the Sum will be found to be 147, which therefore expreſſes the number of Chances for producing XII Points in three Raffles: and in the ſame manner may all the other numbers belonging to the Table of three Raffles be calculated.

This being laid down, let us ſuppoſe that *A* has thrown the Points XL in three Raffles, that there are four Gameſters beſides himſelf, and that under that circumſtance of *A*, it be required to find the Probability of his beating the other four.

Let *m* univerſally repreſent the number of Chances which any other Gameſter has of coming to an equality with *A*, which number of Chances in this particular Caſe is 23886; Let *a* univerſally repreſent the number of Chances which *A* has for beating any one of his Adverſaries, which number of Chances is found in the Table to be 791506; Let *ſ* repreſent the number of all the Chances that there are in three different Raffles, which number is the Cube of 96, by reaſon that there are no more than 96 ſingle Raffles in three Dice, and therefore *ſ* conſtantly ſtands for the number 884736; Let *p* univerſally repreſent the number of Gameſters in all, which in this Caſe will be 5; then the Probability which *A* has of beating the other four will be $\frac{\overline{a+m}\rceil^{p}-a^{p}}{mp\times ſ^{p-1}}$; and therefore if each of the Gameſters ſtake 1, the Expectation of *A* upon the whole Stake *p*, will be expreſſed by $\frac{\overline{a+m}\rceil^{p}-a^{p}}{mſ^{p-1}}$; and conſequently his Gain, or what he might clearly get from his Adverſaries by an equitable compoſition with them for the Value of his Chance, will be $\frac{\overline{a+m}\rceil^{p}-a^{p}}{mſ^{p-1}}-1$.

Now the Logarithm of $\overline{a+m}=5.9113665$, Log. $a=5.8984542$, Log. $m=4.3781434$, Log. $ſ=5.9468136$; and therefore Log. $\overline{a+m}\rceil^{p}=$ or Log. $\overline{a+m}\rceil^{5}=29.5568325$, Log. $a^{p}=29.4922710$, Log. $mſ^{p-1}=28.1653978$; from which Logarithms it will be convenient to reject the leaſt index 28, and treat thoſe Logarithms as if they were reſpectively 1.5568325, 1.4922710, 0.1653978: but the numbers belonging to the two firſt are 36.044 and 31.065, whoſe difference is 4.979 from the Logarithm of which, *viz.* 0.6971421, if the Log. 0.1653978 be ſubtracted, there will remain the Log. 0.5317433, of which the correſponding number being 3.402, it follows that the Gain of *A* ought to be eſtimated by 2.402.

DEMONSTRATION.

1°, When *A* has thrown his Chance, the Probability of *B*'s having a worſe Chance will be $\frac{a}{s}$; wherefore the Probability which *A* has of beating all his Adverſaries whoſe number is $p - 1$, will be $\frac{a^{p-1}}{s^{p-1}}$.

2°, The Probability which *B* has in particular of coming to an Equality with *A* is $\frac{m}{s}$, which being ſuppoſed, the Probability which *A* has of beating the reſt of his Adverſaries whoſe number is $p - 2$, is $\frac{a^{p-2}}{s^{p-2}}$; which being again ſuppoſed, the Probability which *A* now has of beating *B*, with whom he muſt renew the Play, is $\frac{1}{2}$; wherefore the Probability of the happening of all theſe things is $= \frac{m}{s} \times \frac{a^{p-2}}{s^{p-2}} \times \frac{1}{2} = \frac{\frac{1}{2}ma^{p-2}}{s^{p-1}}$: but becauſe *C*, or *D* or *E*, &c. might as well have come to an equality with *A* as *B* himſelf, it follows that the preceding Fraction ought to be multiplied by $p - 1$, which will make it, that the Probability which *A* has of beating all his Adverſaries except one, who comes to an equality with him, and then of his beating him afterwards, will be $\frac{\frac{p-1}{2}ma^{p-2}}{s^{p-1}}$.

3°, The Probability which both *B* and *C* have of coming to an equality with *A* is $\frac{mm}{ss}$; which being ſuppoſed, the Probability which *A* has of beating the reſt of his Adverſaries whoſe number is $p - 3$, is $\frac{a^{p-3}}{s^{p-3}}$; which being again ſuppoſed, the Probability which *A* now has of beating *B* and *C* with whom he muſt renew the Play, (every one of them being now obliged to throw for a new Chance) is $\frac{1}{3}$; wherefore the Probability of the happening of all theſe things will be $= \frac{mm}{ss} \times \frac{a^{p-3}}{s^{p-3}} \times \frac{1}{3} = \frac{\frac{1}{3}mma^{p-3}}{s^{p-1}}$: but the number of the Adverſaries of *A* being $p - 1$, and the different Variations which that number can undergo by elections made two and two being $\frac{p-1}{1} \times \frac{p-2}{2}$, as appears from the Doctrine of Combinations, it follows that the Probability which any two, and no more, of the Adverſaries of *A* have of coming to an Equality with him, that *A* ſhall beat all the reſt, and that he ſhall beat afterwards thoſe two that were come to an Equality, is $\dfrac{\frac{p-1}{2} \times \frac{p-2}{3} mma^{p-3}}{ſ^{p-1}}$ and ſo of the reſt.

From hence it follows that the Probability which *A* has of beating all his Adverſaries, will be expreſſed by the following Series,

$$\frac{a^{p-1} + \frac{p-1}{2} ma^{p-2} + \frac{p-1}{2} \times \frac{p-2}{3} mma^{p-3} + \frac{p-1}{2} \times \frac{p-2}{3} \times \frac{p-3}{4} m^3 a^{p-4}, \&c.}{ſ^{p-1}}$$

the Terms of whoſe Numerator are continued till ſuch time as their number be $= p$; now to thoſe who underſtand how to raiſe a Binomial to a Power given, by means of a Series, it will plainly appear that the foregoing Expreſſion is equivalent to this other $\frac{\overline{a+m}^{\,p} - a^p}{mp \times ſ^{p-1}}$; which conſequently denotes the Probability required.

PROBLEM XLIX.

The ſame things being given as in the preceding Problem, to find how many Gameſters there ought to be in all, to make the Chance of A, *after he has thrown the Point* XL, *to be the moſt advantageous that is poſſible.*

SOLUTION.

It is very eaſily perceived that the more Adverſaries *A* has, the more his Probability of winning will decreaſe; but he has a Compenſation, which is, that if he beats them all, his Gain will be greater than if he had had fewer Competitors: for which reaſon, there being a balance between the Gain that he may make on one ſide, and the decreaſe of the Probability of winning on the other, there is a certain number of Gameſters, which till it be attained, the Gain will be more prevalent than the decreaſe of Probability; but which being exceeded, the decreaſe of Probability will prevail over the Gain; ſo that what was advantage, till a certain time, may gradually turn to equality, and even to diſadvantage. This Problem is therefore propoſed in order to determine thoſe Circumſtances.

Let Log. ſ — Log. a be made $= g$, let alſo Log. ſ — Log. $\overline{a+m}$ be made $= f$, which being done, then the number of Gameſters requiſite to make the Advantage the greateſt poſſible will be expreſſed by the fraction $\frac{\log. g - \log. f}{\log\ \overline{a+m} - \log. a}$, ſo that ſuppoſing as in the preceding Problem that $a = 791506$, $m = 23886$, and conſequently $a + m = 815392$, as alſo ſ $= 884736$, and Log. ſ $= 5.9468136$ Log. $a = 5.8984542$, Log. $m = 4.3781434$ Log. $\overline{a+m} = 5.9113665$, then g will be $= 0.0483594$, and f will be $= 0.354471$. Theref. Log. g — Log. $f = 0.1349014$, and Log. $\overline{a+m}$ — Log. $a = 0.0129123$ and therefore the number of Gameſters will be $\frac{1349014}{129123} = 10.4$ nearly, which ſhews that the number required will be about ten or eleven.

As the Demonſtration of this laſt Operation depends upon principles that are a little too remote from the Doctrine of Chances, I have thought fit to omit it in this place; however if the Reader will be pleaſed to conſult my *Miſcellanea Analytica*, therein he will find it, *pag.* 223 and 224.

It is proper to obſerve, that the method of Solution of this laſt Problem, as well as of the preceding, may be applied to an infinite variety of other Problems, which may happen to be ſo much eaſier than theſe, as they may not require Tables of Chances ready calculated.

PROBLEM L.

Of WHISK.

If four Gameſters play at Whiſk, to find the Odds that any two of the Partners, that are pitched upon, have not the four Honours.

SOLUTION.

Firſt, Suppoſe thoſe two Partners to have the Deal, and the laſt Card which is turned up to be an Honour.

From the Suppoſition of theſe two Caſes, we are only to find what Probability the Dealers have of taking three ſet Cards in twenty-five, out of a Stock containing fifty-one. To reſolve this the ſhorteſt way, recourſe muſt be had to the Theorem given in the Remark belonging to our xxth Problem, in which making the Quantities n, *c*, *d*, *p*, *a*, reſpectively equal to the numbers 51, 25, 26, 3, 3, the Probability required will be found to be $\frac{25\times24\times23}{51\times50\times49}$ or $\frac{92}{833}$.

Secondly, If the Card which is turned up be not an Honour, then we are to find what Probability the Dealers have of taking four given Cards in twenty-five out of a Stock containing fifty-one; which by the aforeſaid Theorem will be found to be $\frac{25\times24\times23\times22}{51\times50\times49\times48}$ or $\frac{253}{4998}$.

But the Probability of taking the four Honours being to be eſtimated before the laſt Card is turned up; and there being ſixteen Chances in fifty-two, or four in thirteen for an Honour to turn up, and nine in thirteen againſt it, it follows that the Probability of the firſt Caſe ought to be multiplied by 4; that the fraction expreſſing the Probability of the ſecond ought to be multiplied by 9; and that the Sum of thoſe Products ought to be divided by 13, which being done, the Quotient $\frac{115}{1666}$ or $\frac{2}{29}$ nearly, will expreſs the Probability required.

And by the ſame Method of proceeding it will be found, that the Probability which the two Eldeſt have of taking four Honours is $\frac{69}{1666}$, that the Probability which the Dealers have of taking three Honours is $\frac{468}{1666}$, and that the Probability which the Eldeſt have of taking three Honours is $\frac{364}{1666}$. Moreover, that the Probability that there are no Honours on either ſide will be $\frac{650}{1666}$.

Hence it may be concluded, 1°, that it is 27 to 2 nearly that the Dealers have not the four Honours.

That it is 23 to 1 nearly that the Eldeſt have not the four Honours.

That it is 8 to 1 nearly that neither one ſide nor the other have the four Honours.

That is 13 to 7 nearly that the two Dealers do not reckon Honours.

That it is 20 to 7 nearly that the two Eldeſt do not reckon Honours.

And that it is 25 to 16 nearly that either one ſide or the other do reckon Honours, or that the Honours are not equally divided.

COROLLARY I.

From what we have ſaid, it will not be difficult to ſolve this Caſe at Whiſk; *viz.* which ſide has the beſt, of thoſe who have VIII of the Game, or of thoſe who at the ſame time have IX?

In order to which it will be neceſſary to premiſe the following Principle.

1°, That there is but 1 Chance in 8192 to get VII by Triks.
2°, That there are 13 Chances in 8192 to get VI.
3°, That there are 78 Chances in 8192 to get V.
4°, That there are 286 Chances in 8192 to get IV.
5°, That there are 715 Chances in 8192 to get III.
6°, That there are 1287 Chances in 8192 to get II.
7°, That there are 1716 Chances in 8192 to get I.

All this will appear evident to thoſe who can raiſe the Binomial $a+b$ to its thirteenth power.

But it muſt carefully be obſerved that the foregoing Chances expreſs the Probability of getting ſo many Points by Triks, and neither more nor leſs.

For if it was required, for Inſtance, to aſſign the Probability of getting one or more by Triks, it is plain that the Numerator of the Fraction expreſſing that Probability would be the Sum of all the Chances which have been written, *viz.* 4096, and conſequently that this Probability would be $\frac{4096}{8192}$ or $\frac{1}{2}$.

2°, That the Probability of getting two or more by Triks would be $\frac{2380}{8192}$, or $\frac{1190}{4090}$.

3°, That the Probability of getting three or more by Triks would be $\frac{1093}{8192}$.

4°, That the Probability of getting IV or more by Triks would be $\frac{378}{8192}$.

5°, That the Probability of getting V or more by Triks would be $\frac{92}{8192}$.

6°, That the Probability of getting VI or more would be $\frac{14}{8192}$.

7°, That the Probability of getting VII would be $\frac{1}{8192}$.

This being laid down, I proceed thus.

1°, If thoſe that have VIII of the Game are Dealers, their Probability of getting II by Honours is $\frac{583}{1660}$: for the Dealers will get II by Honours if they have either 3 of the 4 Honours, or all the 4 Honours, but the Probability of taking three Honours is $\frac{468}{1666}$, and the Probability they have of taking the four Honours is $\frac{115}{1666}$, and the Sum of this is $\frac{583}{1666}$.

The Probability which they have of getting them by Triks is $\frac{2380}{8192}$ or $\frac{1100}{4096}$.

And therefore adding theſe two Probabilities together, the Sum will be $\frac{4370508}{6823936}$.

Now ſubtracting from this, the Probability of both circumſtances happening together, *viz.* $\frac{693770}{6823936}$ the remainder will be $\frac{3676738}{6823936}$; and this expreſſes their Expectation upon the common Stake which we ſuppoſe to be $= 1$.

But they have a farther Expectation, which is that of getting one ſingle Game by Triks, which is $\frac{1716}{8192}$ or $\frac{429}{2048}$; and their Probability of not getting by Honours is $\frac{1083}{1666}$ $\left(= \overline{1 - \frac{583}{1666}}\right)$; and therefore their Probability of getting one ſingle Game by Triks independently from Honours is $\frac{464607}{3411968}$; but then if this happen they will be but equal with their Adverſaries, and therefore this Chance entitles them to no more than half of the common Stake; therefore taking the half of the foregoing fraction, it will be $\frac{464607}{6823936}$; and therefore the whole Expectation of the Dealers is $\frac{3676738 + 464607}{6823936} = \frac{4141345}{6823936}$: whence there remains for thoſe who have IX of the Game $\frac{2682591}{6823936}$; which will make that the Odds for the VIII againſt the IX will be 4141345 to 2682591, which is about 3 to 2, or ſomething more, *viz.* 17 to 11.

2°, But if thoſe who have VIII of the Game are Eldeſt, then their Probability of having three of the four Honours is $\frac{364}{1666}$, and their Probability of having the four Honours is $\frac{69}{1666}$, and therefore their Probability of getting their two Games by Honours is $\frac{364 + 69}{1666} = \frac{433}{1666}$. The Probability of getting them by Triks is as before $\frac{1190}{4096}$, now adding theſe two Probabilities together, the Sum will be $\frac{3756108}{6823936}$, from which ſubtracting, the Probability of both circumſtances happening together, *viz.* $\frac{515270}{6823936}$, there will remain $\frac{3240838}{6823936}$, and this expreſſes the Expectation ariſing from the Proſpect of their winning at once either by Honours or by Triks.

But their Expectation ariſing from the Proſpect of getting one ſingle Game, and then being upon an equal foot with their Adverſaries, found the ſame way as it was in the Suppoſition of their being Dealers, is $\frac{528957}{6823936}$. For the Probability of the Eldeſt taking 4 Honours is $\frac{69}{1666}$, and of their taking 3 Honours, $\frac{364}{1666}$; whoſe Sum taken from Unity, leaves $\frac{1233}{1666}$, for the Probability of their not getting by Honours; and this multiplied by $\frac{429}{2048}$ the Probability of their getting one Game by Triks, gives $\frac{528957}{3411968}$; the half of which is $\frac{528957}{6823936}$. And therefore their Expectation upon the whole is $\frac{3240838+528957}{6823936} = \frac{3769795}{6823936}$, and conſequently there remains for the IX, $\frac{3054141}{6823936}$, and therefore the Odds of the VIII againſt the IX are now 3769795 to 3054141, which is nearly as 95 to 77.

From whence it follows that without conſidering whether the VIII are Dealers or Eldeſt, there is one time with another the Odds of ſomewhat leſs than 7 to 5; and very nearly that of 25 to 18.

COROLLARY 2.

It is a Queſtion likewiſe belonging to this Game, what the Probability is that a Player has a given number of Trumps dealt him: particularly, it has been often taken as an equal Wager that the Dealer has at leaſt 4 Trumps.

Now altho' the Solution of all ſuch Queſtions is included in our xx^th^ Problem; yet as this Game is much in uſe, I have, for the Reader's eaſe, computed the following Tables; ſhewing, for the Dealer as well as the other Gameſters, what the Probability is of taking *preciſely* any aſſigned number of Trumps in one deal.

And thence by a continual addition of the numbers, or of ſuch part of them as is neceſſary, it is eaſily found what the Probability is of taking *at leaſt* that number.

	Chances of the Dealer to have beſides the Card turned up.	Trumps	Chances of any other Gameſter to have preciſely.	
	3910797436	0	8122425444	
	20112672528	I.	46929569232	
	41959196136	II.	110619698904	
	46621329040	III.	139863987120	
	30454255260	IV.	104897990340	
	12181702104	V.	48726808416	
	3014663652	VI.	14211985788	
	455999544	VII.	2583997416	
Tab. I.	40714245	VIII.	284999715	Tab. II.
	2010580	IX.	18095220	
	48906	X.	603174	
	468	XI.	8892	
	1	XII.	39	
	Sum = 158753389900 is the commom Denominator; being the Combinations of 12 Cards in 51.		476260169700 = Sum, is the common Denominator; being the Combinations of 13 in 51.	

By the help of theſe Tables ſeveral uſeful Queſtions may be reſolved; as 1°. If it is aſked, what is the Probability that the Dealer has preciſely III Trumps, beſides the Trump Card? The Anſwer, by *Tab.* I. is $\frac{4662}{15875}$; and the Probability of his having ſome other number of Trumps is $\frac{11213}{15875}$. But if the Queſtion had been, What is the Probability that ſome other Gameſter, the eldeſt hand for inſtance, has preciſely IV Trumps? The anſwer, by *Tab.* II. is $\frac{104898}{476260}$.

2°. To find the Chance of the Dealer's not having fewer than IV Trumps: add his Chances to take 0, I, II, which are 39108, 201127, 419592; and their Sum 659827 taken from the Denominator 1587534, and the Remainder made its Numerator, the Probability of the Dealer having IV or more Trumps will be $\frac{927707}{1587534}$ $=\frac{329}{563}$, a little above $\frac{7}{12}$. The Wager therefore that the Dealer has not IV Trumps is ſo far from equal, that whoever lays it throws away above $\frac{1}{6}$ of his Stake.

But if the Wager is that the Dealer has not V Trumps, then 466213 (the Chances of his having III. beſides the Trump Card) is to be added to the Chances for 0, I, II; which will make the Chance of him who lays this Wager to be nearly $\frac{317}{455}$; and that of his Adverſary $\frac{138}{455}$.

And hence, if Wagers are laid that the Dealer has not IV Trumps, and has not V Trumps, *alternately*; the advantage of him who lays in this manner will be nearly $11\frac{1}{4}$ *per Cent.* of his Stakes.

3°. To find the Odds of laying that the eldeſt hand has at leaſt III, and at leaſt IV Trumps, *alternately*; the Numerator of the one Expectation is (by *Tab.* II.) 31501119, and of the other 17514720, to the Denominator 47626017; whence the advantage of the Bet will be $\frac{15}{514}$, or 3 *per Cent.* nearly.

Again, if it is laid that the Trumps in the Dealer's hand ſhall be either I, II, III or VI; the diſadvantage of this Bet will be only 15$^{ſh.}$ 4^{d}, or about $\frac{3}{4}$, *per Cent.*

In like manner, the Odds of any propoſed Bet of this kind may be computed: And from the Numbers in the Tables, and their Combinations, different Bets may be found which ſhall approach to the Ratio of Equality; or if they differ from it, other Bets may be aſſigned, which, repeated a certain Number of Times, ſhall ballance that difference.

4°, And if the Bet includes any other Condition beſides the number of Trumps, ſuch as the Quality of one or more of them; then proper Regard is to be had to that reſtriction.

Let the Wager be that the Eldeſt has IV Trumps dealt him; and that two of them ſhall be the Ace and King. The Probability of his having IV Trumps preciſely is, by Tab II. $\frac{104808}{476200}$: and the different fours in 12 Cards are $\frac{12}{1} \times \frac{11}{2} \times, \frac{10}{3} \times \frac{9}{4}$. But becauſe 2 out of the 12 Trumps are ſpecified, all the Combinations of 4 in 12 that are favourable to the Wager are reduced to the different two's that are found in the remaining 10 Cards, which are $\frac{10}{3} \times \frac{9}{2}$. And this number is to the former as 1 to 11: the Probability therefore is reduced by this reſtriction to $\frac{1}{11}$, of what elſe it had been: that is, it is reduced from near $\frac{1}{5}$ to about $\frac{1}{52}$.

Note; theſe Tables and others of a like kind, which different Games may require, are beſt computed and examined by beginning with the loweſt number, and obſerving the Law by which the others are formed ſucceſſively. As in Tab. I, putting $A=1$; and the Letters B, C, D, &c. ſtanding for the other Terms regularly aſcending; we ſhall have $B=\frac{39}{1}\times\frac{12}{1}\times A$, $C=\frac{38}{2}\times\frac{11}{2}\times B$, $D=\frac{37}{3}\times\frac{10}{3}\times C$, &c. till we arrive at the Term $N=\frac{28}{12}\times\frac{1}{12}\times M$.

And if the correſponding Terms in Tab. II. are marked by the ſame Letters dotted, then is $A'=\frac{39}{1}\times A$, $B'=\frac{38}{2}\times B$, $C'=\frac{37}{3}\times C$, $D'=\frac{36}{4}\times D$, &c. up to $N'=\frac{27}{13}\times N$.

PROBLEM LI.

Of PIQUET.

To find at Piquet *the Probability which the Dealer has for taking one Ace or more in three Cards, he having none in his Hand.*

SOLUTION.

From the number of all the Cards which are thirty-two, ſubtracting twelve which are in the Dealer's Hands, there remain twenty, among which are the four Aces.

From which it follows that the number of all the Chances for taking any three Cards in the bottom, is the number of Combinations which twenty Cards may afford being taken three and three; which by the Rule given in our XV Problem is $\frac{20\,.\,19\,.\,18}{1\,.\,2\,.\,3}$ or 1140.

The number of all the Chances being thus obtained, find the number of Chances for taking one Ace preciſely with two other Cards; find next the number of Chances for taking two Aces preciſely with any other Card; laſtly, find the number of Chances for taking three Aces; then theſe Chances being added together, and their Sum divided by the whole number of Chances, the Quotient will expreſs the Probability required.

But the number of Chances for taking one Ace are 4, and the number of Chances for taking any two other Cards, are $\frac{16}{1} \cdot \frac{15}{2}$, and therefore the number of Chances for taking one Ace and two other Cards are $\frac{4}{1} \times \frac{16 \cdot 15}{1 \cdot 2} = 480$, as appears from what we have ſaid in the Doctrine of Combinations.

If there remains any difficulty in knowing why the number of Chances for joining any two other Cards with the Ace already taken is $\frac{16 \cdot 15}{1 \cdot 2}$, it will be eaſily reſolved if we conſider that there being in the whole Pack but 4 Aces and 28 other Cards, out of which other Cards, the Dealer has 12 in his Hands, there remain only 16, out of which he has a Choice, and therefore the number of Chances for taking two other Cards is what we have determined.

In like manner it will appear that the number of Chances for taking two Aces preciſely are $\frac{4 \cdot 3}{1 \cdot 2}$ or 6, and that the number of Chances for taking any other Card are $\frac{16}{1}$ or 16; from whence it follows that the number of Chances for taking two Aces with another Card are 6 × 16 or 96.

Laſtly, it appears that the number of Chances for taking three Aces is equal to $\frac{4 \quad 3 \cdot 2}{1 \cdot 2 \cdot 3} = 4$.

Wherefore the Probability required will be found to be $\frac{480+96+4}{1140}$ or $\frac{580}{1140}$ or $\frac{29}{57}$, which fraction being ſubtracted from Unity, the remainder will be $\frac{28}{57}$.

From whence it may be concluded that it is 29 to 28 that the Dealer takes one Ace or more in three Cards, he having none in his Hand.

The preceding Solution may be contracted by inquiring at firſt what the Probability is of not taking any Ace in three Cards, which may be done thus.

The number of Cards in which the four Aces are contained being twenty, and conſequently the number of Cards out of which the four Aces are excluded being ſixteen, it follows that the number of Chances which there are for the taking of three Cards, among which no Ace ſhall be found, is the number of Combinations which ſixteen Cards may afford being taken three and three, which number of Chances by our 18th Problem will be found to be $\frac{16 \cdot 15 \cdot 14}{1 \cdot 2 \cdot 3}$ or 560.

But the number of all the Chances for taking any three Cards in twenty has been found to be 1140; from whence it follows that the Probability of not taking any Ace in three Cards, is $\frac{560}{1140}$ or $\frac{28}{57}$, and therefore the Probability of the contrary, that is of taking one Ace or more in three Cards is $\frac{29}{57}$ as we had found it before.

PROBLEM LII.

To find at Piquet *the Probability which the Eldeſt has of taking an Ace in five Cards, he having no Ace in his Hand.*

SOLUTION.

Firſt, Find the number of Chances for taking one Ace and four other Cards, which will be 7280.

Secondly, The number of Chances for taking two Aces and three other Cards, which will be found to be 3360.

Thirdly, The number of Chances for taking three Aces and two other Cards, which will be found to be 480.

Fourthly, The number of Chances for taking four Aces and any other Card, which will be found to be 16.

Laſtly, The number of Chances for taking any five Cards in twenty, which will be found to be 15504.

Let the Sum of all the particular Chances, *viz.* $7280 + 3360 + 480 + 16$, be divided by the Sum of all the Chances, *viz.* by 15504, and the Quotient will be $\frac{11136}{15504}$ or $\frac{232}{323}$ which being ſubtracted from Unity, the remainder will be $\frac{91}{323}$; and therefore the Odds of the Eldeſt hand taking an Ace or more in five Cards are as 232 to 91, or 5 to 2 nearly.

But if the Probability of not taking an Ace in five Cards be inquired into, the work will be conſiderably ſhortened; for this Probability will be found to be expreſſed by $\frac{16 \times 15 \times 14 \cdot 13 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$ or

4368

4368 to be divided by the whole number of Chances, *viz.* by 15504, or 91 by 323; which makes the Probability of taking one or more Aces $\frac{232}{323}$ as before.

PROBLEM LIII.

To find at Piquet *the Probability which the Eldeſt has of taking both an Ace and a King in five Cards, he having none in his Hand.*

SOLUTION.

Let the following Chances be found; *viz.*

1°, For one Ace, one King, and three other Cards.
2°, For one Ace, two Kings, and two other Cards.
3°, For one Ace, three Kings, and any other Card.
4°, For one Ace, and four Kings.
5°, For two Aces, one King, and two other Cards.
6°, For two Aces, two Kings, and any other Card.
7°, For two Aces, and three Kings.
8°, For three Aces, one King, and any other Card.
9°, For three Aces, and two Kings.
10°, For four Aces, and one King.

Among theſe Caſes, there being four pairs that are alike, *viz.* the ſecond and fifth, the third and eighth, the fourth and tenth, the ſeventh and ninth; it follows that there are only ſix Caſes to be calculated, whereof the firſt and ſixth are to be taken ſingly, but the ſecond, third, fourth and ſeventh to be doubled; now the Operation is as follows.

The *firſt* Caſe has $\frac{4}{1}\times\frac{4}{1}\times\frac{12\,.\,11\,.\,10}{1\,.\,2\,.\,3}$ or 3520 Chances.

The *ſecond* $\frac{4}{1}\times\frac{4\,.\,3}{1\,.\,2}\times\frac{12\,.\,11}{1\,.\,2}$ or 1584, the double of which is 3168.

The *third* $\frac{4}{1}\times\frac{4\,.\,3\,.\,2}{1\,.\,2\,.\,3}\times\frac{12}{1}$ or 192, the double of which is 384 Chances.

The *fourth* $\frac{4}{1} \times \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4}$ or 4, the double of which is 8 Chances.

The *ſixth* $\frac{4 \cdot 3}{1 \cdot 2} \times \frac{4 \cdot 3}{1 \cdot 2} \times \frac{12}{1}$ or 432 Chances.

The *ſeventh* $\frac{4 \cdot 3}{1 \cdot 2} \times \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} \times$ or 24, the double of which is 48 Chances.

Now the Sum of all thoſe Chances being 7560, and the whole number of Chances for taking any five Cards out of 20 being $\frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$ or 15504, it follows that the Probability required will be $\frac{7560}{15504}$ or $\frac{315}{646}$, and therefore the Probability of the contrary will be $\frac{331}{646}$, from whence it follows that the Odds againſt the Eldeſt hand taking an Ace and a King are 331 to 315, or 21 to 20 nearly.

PROBLEM LIV.

To find at Piquet the Probability of having twelve Cards dealt to, without King, Queen or Knave, which Caſe is commonly called Cartes Blanches.

SOLUTION.

Altho' this may be derived from what has been ſaid in the xx$^{\text{th}}$ Problem, yet I ſhall here preſcribe a Method which will be ſomewhat more eaſy, and which may be followed in many other Inſtances.

Let us therefore imagine that the twelve Cards dealt to are taken up one after another, and let us conſider, 1°, the Probability of the firſt's being a Blank; now there being 20 Blanks in the whole Pack, and 32 Cards in all, it is plain that the Probability of it is $\frac{20}{32}$. 2°, Let us conſider the Probability of the ſecond's being a Blank, which by reaſon the firſt Card is accounted for, and becauſe, there remain now but 19 Blanks and 31 Cards in all, will be found to be $\frac{19}{31}$; and in like manner the Probability of the third Card's being a Blank will be $\frac{18}{30}$, and ſo on; and therefore the

Probability of the whole will be expreſſed by the Fraction $\frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{32 \cdot 31 \cdot 30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21}$ the number of Multiplicators in both Numerator and Denominator being equal to twelve. Now that Fraction being ſhortened will be reduced to $\frac{323}{578950}$ or $\frac{1}{1792}$ nearly, and therefore the Odds againſt *Cartes Blanches* are 1791 to 1 nearly.

PROBLEM LV.

To find how many different Sets, eſſentially different from one another, one may have at Piquet before taking in.

SOLUTION.

Let the Suits be diſpoſed in order, and let the various diſpoſitions of the Cards be written underneath, together with the number of Chances that each diſpoſition will afford, and the Sum of all thoſe Chances will be the thing required.

Let alſo the Letters D, H, S, C reſpectively repreſent Diamonds, Hearts, Spades, and Clubs.

	D,	*H,*	*S,*	*C*	*Chances.*
1	0,	0,	4,	8 = ...	70
2	0,	0,	5,	7 =	448
3	0,	0,	6,	6 =	748
4	0,	1,	3,	8 =	448
5	0,	1,	4,	7 =	4480
6	0,	1,	5,	6 =	12544
7	0,	2,	2,	8 =	784
8	0,	2,	3,	7 =	812544
9	0,	2,	4,	6 =	54880
10	0,	2,	5,	5 =	87808
11	0,	3,	3,	6 =	87808
12	0,	3,	4,	5 =	219520
13	0,	4,	4,	4 =	343000
14	1,	1,	2,	8 =	1792
15	1,	1,	3,	7 =	28672
16	1,	1,	4,	6 =	125440
17	1,	1,	5,	5 =	200704
18	1,	2,	2,	7 =	50176
19	1,	2,	3,	6 =	351232
20	1,	2,	4,	5 =	878080
21	1,	3,	3,	5 =	1404928
22	1,	3,	4,	4 =	2195200
23	2,	2,	2,	6 =	614656
24	2,	2,	3,	5 =	2458624
25	2,	2,	4,	4 =	3851600
26	2,	3,	3,	4 =	6146560
27	3,	3,	3,	3 =	9834496
	Sum				28,967,278

Which Sum would ſeem incredibly great, if Calculation did not prove it to be ſo.

But it will not be inconvenient to ſhew by one Example how the numbers expreſſing the Chances have been found, for which we muſt have recourſe to our XX^th^ and XXI^th^ Problems, and there examine the Method of Solution, the ſame being to be obſerved in this place. Let it therefore be required to aſſign the 19^th^ Caſe, which is for taking 1 Diamond, 2 Hearts, 3 Spades and 6 Clubs. Then it will eaſily be ſeen that the variations for taking 1 Diamond are 8, that the variations for taking 2 Hearts are $\frac{8\,.\,7}{1\,.\,2} = 28$, and that the variations for taking 3 Spades are $\frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$, and that the variations for taking 6 Clubs are $\frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = 28$. And therefore that the number of Chances for the 19th Caſe is the product of the ſeveral numbers 8, 28, 56, 28, which will be found 351232.

There is one thing worth obſerving, which is, that when the number of Cards of any one Suit being to be combined together, exceed one half the number of Cards of that Suit, then it will be ſufficient to combine only the difference between that number and the whole number of Cards in the Suit, which will make the operation ſhorter; thus being to combine the 8 Clubs by ſix and ſix, I take the difference between eight and ſix, which being 2, I combine the Cards only two and two, it being evident that as often as I take 6 Cards of one Suit, I leave 2 behind of the ſame Suit, and that therefore I cannot take them oftner ſix and ſix, than I can take them two and two.

It may perhaps ſeem ſtrange that the number of Sets which we have determined, notwithſtanding its largeneſs, yet ſhould not come up to the number of different Combinations whereby twelve Cards might be taken out of thirty-two, that number being 225792840; but it ought to be conſidered, that in that number ſeveral ſets of the ſame import, but differing in Suit might be taken, which would not introduce an eſſential difference among the Sets.

REMARK.

It may eaſily be perceived from the Solution of the preceding Problem, that the number of variations which there are in twelve Cards make it next to impoſſible to calculate ſome of the Probabilities relating to Piquet, ſuch as that which reſults from the priority of Hand, or the Probabilities of a Pic, Repic or Lurch; however notwithſtanding that difficulty, one may from obſervations often repeated, nearly eſtimate what thoſe Probabilities are in themſelves, as will be proved in its place when we come to treat of the reaſonable conjectures which may be deduced from Experiments; for which reaſon I ſhall ſet down ſome Obſervations of a Gentleman who has a very great degree of Skill and Experience in that Game, after which I ſhall make an application of them.

HYPOTHESES.

1°, That 'tis 5 to 4 that the Eldeſt hand wins a Game.

2°,

2°, That is 2 to 1, that the Eldeſt wins rather without lurching than by lurching.

3°, That it is 4 to 1, that the Youngeſt Hand wins rather without lurching than by lurching.

But it muſt carefully be obſerved that theſe Odds are reſtrained to the beginning of a Game.

From whence, to avoid Fractions, we may ſuppoſe that the Eldeſt has 75 Chances to win one Game, and the Youngeſt 60.

That out of theſe 75 Chances of the Eldeſt, he has 50 to win without Lurch, and 25 with a Lurch.

That of the 60 Chances of the Youngeſt, he has 48 to win without a Lurch, and 12 with a Lurch.

This being laid down, I ſhall proceed to determine the Probabilities of winning the Set, under all the circumſtances in which *A* and *B* may find themſelves.

1°, When *A* and *B* begin, he who gets the Hand has the Odds of 6478643 to 3362857 or 23 to 20 nearly that he wins the Set.

2°, If *A* has 1 Game and *B* none.

Before they cut for the Hand, the Odds in favour of *A* are 682459 to 309067 or 38 to 23 nearly.

If *A* has the Hand, the Odds are 4627 to 1448, or 16 to 5 nearly.

If *B* has the Hand, the Odds in favour of *A* are 511058 to 309067, or 38 to 23 nearly.

3°, If *A* has 1 Game, and *B* 1 Game.

He who gets the Hand has the Odds of 10039 to 8186 or 27 to 22 nearly.

4°, If *A* has 2 Games and *B* none.

Before they cut for the Hands the Odds are 59477 to 13423, or 31 to 7 nearly.

If *A* has the Hand, the Odds are 5117 to 958, or 16 to 3 nearly.

If *B* has the Hand, the Odds in favour of *A* are 1151 to 307, or 25 to 7 nearly.

5°, If *A* has 2 Games and *B* 1.

Before they cut for the Hand, the Odds are 92 to 43, or 15 to 7 nearly.

If *A* has the Hand, the Odds are 11 to 4. *

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* In this Caſe *B* has 12 Chances for 1, and 48 for $\frac{1}{2}$, but the number of all the Chances

If *B* has the Hand, the Odds in favour of *A* are 17 to 10.

6°, If *A* has 2 Games and *B* 2 Games, he who gets the Hand has 5 to 4 in his favour.

I hope the Reader will eaſily excuſe my not giving the Demonſtration of the foregoing Calculation, it being ſo eaſily deduced from the Rules given before, that this would ſeem entirely ſuperfluous.

PROBLEM LVI.

Of SAVING CLAUSES.

A *has* 2 *Chances to beat* B, *and* B *has* 1 *Chance to beat* A; *but there is one Chance which intitles them both to withdraw their own Stake, which we ſuppoſe equal to* ſ; *to find the Gain of* A.

SOLUTION.

This Queſtion tho' eaſy in itſelf, yet is brought in to caution Beginners againſt a Miſtake which they might commit by imagining that the Caſe, which intitles each Man to recover his own Stake, needs not be regarded, and that it is the ſame thing as if it did not exiſt: This I mention ſo much more readily, that ſome people who have pretended great ſkill in theſe Speculations of Chance have themſelves fallen into that error. Now there being 4 Chances in all, whereof *A* has 2 to gain ſ, 'tis evident that the Expectation of that Gain is worth $\frac{2}{4}$ſ; but *A* having 1 Chance in 4 to loſe ſ, the Riſk of that is a Loſs which muſt be eſtimated by $\frac{1}{4}$ſ, and therefore the abſolute Gain of *A* is $\frac{2}{4}$ſ $-$ $\frac{1}{4}$ſ, or $\frac{1}{4}$ſ. But ſuppoſing the ſaving Clauſe not conſidered, *A* would have 2 Chances in 3 to win ſ, and 1 Chance in 3 to loſe ſ, and therefore the Expectation of his Gain

---

Chances between *A* and *B* are 135, therefore *B* has $\frac{12+24}{135} = \frac{36}{135} = \frac{4}{15}$, Odds 11 to 4. If *B* has the Hand, then he has 25 for 1, 50 for $\frac{1}{2} = \frac{25+25}{135} = \frac{50}{135} = \frac{10}{27}$, Odds 17 to 10. But before they cut for the Hand *B* has $\frac{4}{15} + \frac{10}{27} + \frac{1}{2} = \frac{43}{135}$, Odds 92 to 43.

would be worth $\frac{2}{3}ſ$, and the Riſk of his Loſs would be eſtimated by $\frac{1}{3}ſ$; which would make his Gain to be $\frac{2}{3}ſ - \frac{1}{3}ſ = \frac{1}{3}ſ$. From whence it may evidently be ſeen that the condition of drawing Stakes is to be conſidered; and indeed in this laſt Caſe, there are the Odds of 2 to 1 that *A* beats *B*, whereas in the former it cannot be ſaid but very improperly that *A* has 2 to 1 the beſt of the Game; for if *A* undertakes without any limitation to beat *B*, then he muſt loſe if the ſaving Clauſe happens, and therefore he has but an equality of Chance to beat or not to beat; however it may be ſaid with ſome propriety of Expreſſion, that it is 2 to 1 that *A* rather beats *B* than that *A* beats him.

But to make the Queſtion more general, let *A* and *B* each depoſite the Sum $ſ$; let a repreſent the Chances which *A* has to beat *B*, and b the Chances which *B* has to beat *A*; let there be alſo a certain number m of Chances which may be called common, by the happening of which *A* ſhall be entitled to take up ſuch part of the common Stake $2ſ$ as may be denominated by the fraction $\frac{p}{r}$, and *B* ſhall be entitled to take the remainder of it.

Then 1°, it appears that the number of all the Chances being $a+b+m$, whereof there are the number a which intitle *A* to gain $ſ$; thence his Gain upon that ſcore is $\frac{a}{a+b+m} \times ſ$.

2°, It appears that the number of Chances whereby *A* may loſe, being b, his Loſs upon that account is $\frac{b}{a+b+m} \times ſ$.

3°, It appears that if the Chances m ſhould happen, then *A* would take up the part $\frac{p}{r}$ of the common Stake $2ſ$, and thereby gain $\frac{2p}{r}ſ - ſ$, or $\frac{2p-r}{r} \times ſ$. But the Probability of the happening of this is $\frac{m}{a+b+m}$; and therefore his Gain ariſing from the Probability of this circumſtance is $\frac{m}{a+b+m} \times \frac{\overline{2p-r}}{r} \times ſ$.

From all which it appears that his abſolute Gain is $\frac{a}{a+b+m} \times ſ - \frac{b}{a+b+m} \times ſ + \frac{m}{a+b+m} \times \frac{\overline{2p-r}}{r} \times ſ$.

Now ſuppoſe there had been no common Chances, the Gain of *A* would have been $\frac{a-b}{a+b} \times ſ$.

Let it therefore be farther required to aſſign what the proportion of p to r ought to be, to make the Gain of *A* to be the ſame in both Caſes.

This will be eaſily done by the Equation $\frac{a-b}{a+b+m} + \frac{2pm-rm}{r\times\overline{a+b+m}} = \frac{a-b}{a+b}$; wherein multiplying all the Terms by $a+b+m$ we ſhall have the new Equation $a-b+\frac{2pm-rm}{r} = \frac{aa+am-bb-bm}{a+b}$ or $\frac{2pm-rm}{r} = \frac{am-bm}{a+b}$, or $\frac{2p-r}{r} = \frac{a-b}{a+b}$, or $2pa - ra + 2bp - br = ra - br$, or $2pa + 2bp = 2ra$, and therefore $pa + bp = ra$, and $\frac{p}{r} = \frac{a}{a+b}$. From which we may conclude, that if the two parts of the common Stake 2ſ which *A* and *B* are reſpectively to take up, upon the happening of the Chances *m*, are reſpectively in the proportion of *a* to *b*, then the common Chances give no advantage to *A* above what he would have had if they had not exiſted.

PROBLEM LVII.

Odds of Chance and Odds of Money compared.

A *and* B *playing together depoſit* ſ$^{L.}$ *apiece*; A *has* 2 *Chances to win* ſ, *and* B 1 *Chance to win* ſ, *whereupon* A *tells* B *that he will play with him upon an equality of Chance, if he* B *will ſet him* 2ſ *to* 1ſ, *to which* B *aſſents*: *to find whether* A *has any advantage or diſadvantage by that Bargain.*

SOLUTION.

In the firſt circumſtance, *A* having 2 Chances to win *ſ*, and 1 Chance to loſe *ſ*, his Gain, as may be deduced from the Introduction, is $\frac{2ſ-ſ}{3} = \frac{1}{3}ſ$.

In the ſecond circumſtance, *A* having 1 Chance to win 2*ſ*, and 1 Chance to loſe *ſ*, his Gain is $\frac{2ſ-ſ}{2} = \frac{1}{2}ſ$, and therefore he gets $\frac{1}{6}ſ$ by that Bargain.

But if *B*, after the Bargain propoſed, ſhould anſwer, let us play upon an equality of Chance, and you ſhall ſtake but $\frac{1}{2}ſ$, and I ſhall ſtake *ſ*, and ſo I ſhall have ſet 2 to 1, and that *A* ſhould aſſent: then he has 1 Chance to win *ſ*, and 1 Chance to loſe $\frac{1}{2}ſ$, and therefore his Gain is $\frac{s-\frac{1}{2}s}{2}=\frac{1}{4}s$, and therefore he is worſe by $\frac{1}{12}s$ than he was in the firſt circumſtance.

But if *A*, after this propoſal of *B*, anſwers; let us preſerve the quantity of the whole Stake $2s$, but do you ſtake $\frac{4}{3}s$, and I ſhall ſtake $\frac{2}{3}s$, whereby the proportion of 2 to 1 will remain, and that *B* aſſents; then *A* has 1 Chance to win $\frac{4}{3}s$ and 1 Chance to loſe $\frac{2}{3}s$, which makes his Gain to be $\frac{\frac{4}{3}s-\frac{2}{3}s}{2}=\frac{2}{3}s-\frac{1}{3}s=\frac{1}{3}s$, which is the ſame as in the firſt circumſtance.

And univerſally, *A* having a Chances to win s, and *B* having b Chances to win s, if they ſhould agree afterwards to play upon an equality of Chance, and ſet to each other the reſpective Stakes $\frac{2b}{a+b}s$ and $\frac{2a}{a+b}s$, then the Gain of *A* would thereby receive no alteration, it being in both Caſes $\frac{a-b}{a+b}s$.

PROBLEM LVIII.

OF THE DURATION OF PLAY.

Two Gameſters A *and* B *whoſe proportion of ſkill is as* a *to* b, *each having a certain number of Pieces, play together on condition that as often as* A *wins a Game,* B *ſhall give him* one *Piece; and that as often as* B *wins a Game,* A *ſhall give him* one *Piece; and that they ceaſe not to play till ſuch time as either one or the other has got all the Pieces of his Adverſary: now let us ſuppoſe two Spectators* R *and* S *concerning themſelves about the ending of the Play, the firſt of them laying that the Play will be ended in a certain number of Games which he aſſigns, the other laying to the contrary. To find the Probability that* S *has of winning his wager.*

SOLU-

SOLUTION.

This Problem having ſome difficulty, and it having given me occaſion to inquire into the nature of ſome Series naturally reſulting from its Solution, whereby I have made ſome improvements in the Method of ſumming up Series, I think it neceſſary to begin with the ſimpleſt Caſes of this Problem, in order to bring the Reader by degrees to a general Solution of it.

CASE I.

Let 2 be the number of Pieces, which each Gameſter has; let alſo 2 be the number of Games about which the Wager is laid: now becauſe 2 is the number of Games contended for, let $a+b$ be raiſed to its Square, *viz.* $aa+2ab+bb$; then it is plain that the Term $2ab$ favours *S*, and that the other two are againſt him; and conſequently that the Probability he has of winning is $\frac{2ab}{\overline{a+b}^{2}}$.

COROLLARY

If a and b are equal, neither *R* or *S* have any Advantage or Diſadvantage; but if a and b are unequal, *R* has the Advantage.

CASE II.

Let 2 be the number of Pieces of each Gameſter, as before, but let 3 be the number of Games about which the Wager is laid: then $a+b$ being raiſed to its Cube, *viz.* $a^3+3aab+3abb+b^3$, it will be ſeen that the two Terms a^3 and b^3 are contrary to *S*, they denoting the number of Chances for winning three times together; it will alſo be ſeen that the other two Terms $3aab$ and $3abb$ are partly for him, partly againſt him. Let therefore thoſe two Terms be divided into their proper parts, *viz.* $3aab$ into $aab+aba+baa$, and $3abb$ into $abb+bab+bba$, and it will plainly be perceived that out of thoſe ſix parts there are four which are favourable to *S*, *viz.* aab, baa, abb, bba or $2aab+2abb$; from whence it follows that the Probability which *S* has of winning his Wager will be $\frac{2aab+2abb}{\overline{a+b}^{3}}$, or dividing both Numerator and Denominator by $a+b$, it will be found to be $\frac{2ab}{\overline{a+b}^{2}}$, which is the ſame as in the preceding Caſe. The reaſon of which is, that the winning of a certain number of even Pieces in an odd number of Games is impoſſible, unleſs it was done in the even number of Games immediately preceding the odd number, no more than an odd number of Pieces can be won in an even number of Games, unleſs it was done in the odd number immediately preceding it; but ſtill the Problem of winning an even number of Pieces in an odd number of Games is rightly propoſed; for Inſtance, the Probability of winning either of one ſide or the other, 8 Pieces in 63 Games; for, provided it be done either before or at the Expiration of 62 Games, he who undertakes that it ſhall be done in 63 wins his Wager.

CASE III.

Let 2 be the number of Pieces of each Gameſter, and 4 the number of Games upon which the Wager is laid: let therefore $a+b$ be raiſed to the fourth Power, which is $a^4 + 4a^3b + 6aabb + 4ab^3 + b^4$; which being done, it is plain that the Terms $a^4 + 4a^3b + 4ab^3 + b^4$ are wholly againſt S, and that the only Term $6aabb$ is partly for him, and partly againſt him, for which reaſon, let this Term be divided into its parts, *viz.* *aabb*, *abab*, *abba*, *baab*, *baba*, *bbaa*, and 4 of theſe parts, *viz.* *abab*, *abba*, *baab*, *baba*, or $4aabb$ will be found to favour S; from which it follows that his Probability of winning will be $\frac{4aabb}{\overline{a+b}|^4}$.

CASE IV.

If 2 be the number of Pieces of each Gameſter, and 5 the number of Games about which the Wager is laid, the Probability which S has of winning his wager will be the ſame as in the preceding Caſe, *viz.* $\frac{4aabb}{\overline{a+b}|^4}$.

Univerſally, Let 2 be the number of Pieces of each Gameſter, and $2+d$ the number of Games upon which the Wager is laid; and the Probability which S has of winning will be $\frac{\overline{2ab}|^{1+\frac{1}{2}d}}{\overline{a+b}|^{2+d}}$ if d be an even number; or $\frac{\overline{2ab}|^{\frac{1+d}{2}}}{\overline{a+b}|^{1+d}}$ if d be odd, writing $d-1$ inſtead of d.

CASE V.

If 3 be the number of Pieces of each Gameſter, and $3+d$ the number of Games upon which the Wager is laid, then the Probability which S has of winning will be $\dfrac{\overline{3ab}\rceil^{1+\frac{1}{2}d}}{\overline{a+b}\rceil^{2+d}}$ if d be an even number, or $\dfrac{\overline{3ab}\rceil^{\frac{1+d}{2}}}{\overline{a+b}\rceil^{1+d}}$ if it be odd.

CASE VI.

If the number of Pieces of each Gameſter be more than 3, the Expectation of S, or the Probability there is that the Play ſhall not be ended in a given number of Games, may be determined in the following manner.

A General Rule for determining what Probability there is that the Play ſhall not be determined in a given number of Games.

Let n be the number of Pieces of each Gameſter. Let alſo $n+d$ be the number of Games given; raiſe $a+b$ to the Power n, then cut off the two extream Terms, and multiply the remainder by $aa+2ab+bb$: then cut off again the two Extreams, and multiply again the remainder by $aa+2ab+bb$, ſtill rejecting the two Extreams; and ſo on, making as many Multiplications as there are Units in $\frac{1}{2}d$; make the laſt Product the Numerator of a Fraction whoſe Denominator let be $\overline{a+b}\rceil^{n+d}$, and that Fraction will expreſs the Probability required, or the Expectation of S upon a common Stake 1, ſuppoſed to be laid between R and S; ſtill obſerving that if d be an odd number, you write $d-1$ in its room.

EXAMPLE I.

Let 4 be the number of Pieces of each Gameſter, and 10 the number of Games given: in this Caſe $n=4$, $n+d=10$; wherefore $d=6$, and $\frac{1}{2}d=3$. Let therefore $a+b$ be raiſed to the fourth Power, and rejecting continually the extreams, let three Multiplications be made by $aa+2ab+bb$. Thus,

$$
\begin{array}{llllll}
a^4 \,| & +4a^3b & +6aabb & +4ab^3 \,| & +b^4 & \\
 & aa & +2ab & +bb & & \\
\hline
4a^5b \,| & +6a^4bb & +4a^3b^3 & & & \\
 & +8a^4bb & +12a^3b^3 & +8aab^4 & & \\
 & & +4a^3b^3 & +6aab^4 \,| & +4ab^5 & \\
\hline
\end{array}
$$

$14a^4bb$

$$\begin{array}{llllll}
14a^4bb & +20a^3b^3 & +14aab^4 & & & \\
aa & +2ab & +bb & & & \\
\hline
14^6abb| & +20a^5b^3 & +14a^4b^4 & & & \\
 & +28a^5b^3 & +40a^4b^4 & +28a^3b^5 & & \\
 & & +14a^4b^4 & +20a^3b^5| & +14aab^6 & \\
\hline
 & 48a^5b^3 & +68a^4b^4 & +48a^3b^5 & & \\
 & aa & +2ab & +bb & & \\
\hline
 & 48a^7b^3| & +68a^6b^4 & +48a^5b^5| & & \\
 & & +96a^6b^4 & +136a^5b^5 & +96a^3b^6 & \\
 & & & +48a^5b^5 & +68a^4b^6| & +48a^4b^7 \\
\hline
 & & 164a^6b^4 & +232a^5b^5 & +164a^4b^6 &
\end{array}$$

Wherefore the Probability that the Play will not be ended in 10 Games will be $\frac{164a^6b^4+232a^5b^5+164a^4b^6}{\overline{a+b}|^{16}}$, which Expreſſion will be reduced to $\frac{560}{1024}$, if there be an equality of Skill between the Gameſters; now this Fraction $\frac{560}{1024}$ or $\frac{35}{64}$ being ſubtracted from Unity, the remainder will be $\frac{29}{64}$, which will expreſs the Probability of the Play's ending in 10 Games, and, conſequently it is 35 to 29 that, if two equal Gameſters play together, there will not be four Stakes loſt on either ſide, in 10 Games.

N. B. The foregoing operation may be very much contracted by omitting the Letters a and b, and reſtoring them after the laſt Multiplication; which may be done in this manner. Make $n+\frac{1}{2}d-1=p$, and $\frac{1}{2}d+1=q$; then annex to the reſpective Terms reſulting from the laſt Multiplication the literal Products a^pb^q, $a^{p-1}b^{q+1}$, $a^{p-2}b^{q+2}$, &c.

Thus in the foregoing Example, inſtead of the firſt Multiplicand $4a^3b+6aabb+4ab^3$, we might have taken only $4+6+4$, and inſtead of multiplying three times by $aa+2ab+bb$, we might have multiplied only by $1+2+1$, which would have made the laſt Terms to have been $164+232+164$. Now ſince that $n=4$ and $d=6$, p will be $=6$ and $q=4$, and conſequently the literal Products to be annexed reſpectively to the Terms $164+232+164$ will be a^6b^4, a^5b^5, a^4b^6, which will make the Terms reſulting from the laſt Multiplication to be $164a^6b^4+232a^5b^5+164a^4b^6$, as they had been found before.

EXAMPLE II.

Let 5 be the number of Pieces of each Gameſter, and 10 the number of Games given: let alſo the proportion of Skill between *A* and *B* be as 2 to 1.

Since $n = 5$, and $n + d = 10$, it follows that $d = 5$. Now d being an odd number muſt be ſuppoſed $= 4$, ſo that $\frac{1}{2}d = 2$: let therefore $1 + 1$ be raiſed to the fifth Power, and always rejecting the Extreams, multiply twice by $1 + 2 + 1$, thus

$$\begin{array}{l}
1| + 5 + 10 + 10 + 5| + 1 \\
\underline{\;\;1 + 2 + 1\qquad\qquad\qquad} \\
\;\;5| + 10 + 10 + 5 \\
\qquad + 10 + 20 + 20 + 10 \\
\underline{\qquad\qquad + 5 + 10 + 10| + 5} \\
\qquad 20 + 35 + 35 + 20
\end{array}
\qquad\qquad
\begin{array}{l}
20 + 35 + 35 + 20 \\
\underline{\qquad 1 + 2 + 1\qquad\qquad} \\
20| + 35 + 35 + 20 \\
\qquad + 40 + 70 + 70 + 40 \\
\underline{\qquad\qquad + 20 + 35 + 35| + 20} \\
\qquad 75 + 125 + 125 + 75
\end{array}$$

Now to ſupply the literal Products that are wanting, let $n + \frac{1}{2}d - 1$ be made $= p$, and $\frac{1}{2}d + 1 = q$, and the Products that are to be annexed to the numerical quantities will be $a^p b^q$, $a^{p-1} b^{q+1}$, $a^{p-2} b^{q+2}$, $a^{p-3} b^{q+3}$, &c. wherefore n, in this Caſe, being $= 5$, and $d = 4$, then p will be $= 6$, and $q = 3$, it follows that the Products to be annexed in this Caſe be a^5b^3, a^5b^4, a^4b^5, a^3b^6, and conſequently the Expectation of S will be found to be $\frac{75a^6b^3 + 125a^5b^4 + 125a^4b^5 + 75a^3b^6}{\overline{a+b}|^9}$.

N. B. When n is an odd number, as it is in this Caſe, the Expectation of S will always be diviſible by $a + b$. Wherefore dividing both Numerator and Denominator by $a + b$, the foregoing Expreſſion will be reduced to

$$\frac{75a^5b^3 + 50a^4b^4 + 75a^3b^5}{\overline{a+b}|^8} \text{ or } 25a^3b^3 \times \frac{3aa + 2ab + 3bb}{\overline{a+b}|^8}$$

Let now a be interpreted by 2, and b by 1, and the Expectation of S will become $\frac{3800}{6561}$.

PRO-

PROBLEM LIX.

The ſame things being given as in the preceding Problem, to find the Expectation of R, *or otherwiſe the Probability that the Play will be ended in a given number of Games.*

Solution.

Firſt, It is plain that if the Expectation of *S* obtained by the preceding Problem be ſubtracted from Unity, there will remain the Expectation of *R*.

Secondly, Since the Expectation of *S* decreaſes continually, as the number of Games increaſes, and that the Terms we rejected in the former Problem being divided by $aa + 2ab + bb$ are the Decrement of his Expectation; it follows that if thoſe rejected Terms be divided continually by $aa + 2ab + bb$ or $\overline{a+b}^{\,2}$, they will be the Increment of the Expectation of *R*. Wherefore the Expectation of *R* may be expreſſed by means of thoſe rejected Terms. Thus in the ſecond Example of the preceding Problem, the Expectation of *R* expreſſed by means of the rejected Terms will be found to be

$$\frac{a^5+b^5}{\overline{a+b}^{\,5}} + \frac{5a^6b+5ab^6}{\overline{a+b}^{\,7}} + \frac{20a^7bb+20aab^7}{\overline{a+b}^{\,9}}, \text{ or}$$

$$\frac{a^5+b^5}{\overline{a+b}^{\,5}} \times \overline{1 + \frac{5ab}{\overline{a+b}^{\,2}} + \frac{20aabb}{\overline{a+b}^{\,4}}}$$

In like manner, if 6 were the number of the Pieces of each Gameſter, and the number of Games were 14, it would be found that the Expectation of *R* would be

$$\frac{a^6+b^6}{\overline{a+b}^{\,6}} \times \overline{1 + \frac{6aa}{\overline{a+b}^{\,2}} + \frac{27aabb}{\overline{a+b}^{\,4}} + \frac{110a^3b^3}{\overline{a+b}^{\,6}} + \frac{429}{\overline{a+b}^{\,8}}\, a^4b^4}.$$

And if 7 were the number of Pieces of each Gameſter, and the number of Games were 15, then the Expectation of *R* would be found to be

$$\frac{a^7+b^7}{\overline{a+b}^{\,7}} \times \overline{1 + \frac{7ab}{\overline{a+b}^{\,2}} + \frac{35aabb}{\overline{a+b}^{\,4}} + \frac{154a^3b^3}{\overline{a+b}^{\,6}} + \frac{637a^4b^4}{\overline{a+b}^{\,8}}}.$$

N. B. The number of Terms of theſe Series will always be equal to $\frac{1}{2}d + 1$, if d be an even number, or to $\frac{d+1}{2}$, if it be odd.

Thirdly, All the Terms of theſe Series have to one another certain Relations, which being once diſcovered, each Term of any Series reſulting from any Caſe of this Problem, may be eaſily generated from the preceding ones.

Thus in the firſt of the two laſt foregoing Series, the numerical Coefficient belonging to the Numerator of each Term may be derived from the preceding, in the following manner. Let K, L, M be the three laſt Coefficients, and let N be the Coefficient of the next Term required; then it will be found that N in that Series will conſtantly be equal to $6M - 9L + 2K$. Wherefore if the Term which would follow $\frac{429a^4b^4}{\overline{a+b}\vert^{4}}$ in the Caſe of 16 Games given, were deſired; then make $M = 429$, $L = 110$, $K = 27$, and the following Coefficient will be found 1638. From whence it appears that the Term itſelf would be $\frac{1638a^5b^5}{\overline{a+b}\vert^{10}}$.

Likewiſe, in the ſecond of the two foregoing Series, if the Law by which each Term is related to the preceding were demanded, it might thus be found. Let K, L, M be the Coefficients of the three laſt Terms, and N the Coefficient of the Term deſired; then N will in that Series conſtantly be equal to $7M - 14L + 7K$, or $\overline{M - 2L + K} \times 7$. Now this Coefficient being obtained, the Term to which it belongs is formed immediately.

But if the univerſal Law by which each Coefficient is generated from the preceding be demanded, it will be expreſſed as follows.

Let n be the number of Pieces of each Gameſter: then each Coefficient contains

n times the laſt
$- n \times \frac{n-3}{2}$ times the laſt but one
$+ n \times \frac{n-1}{2} \times \frac{n-5}{3}$ times the laſt but two
$- n \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n-7}{4}$ times the laſt but three
$+ n \times \frac{n-6}{2} \times \frac{n-7}{3} \times \frac{n-8}{4} \times \frac{n-9}{5}$ times the laſt but four.
&c.

Thus the number of Pieces of each Gameſter being 6, the firſt Term n would be $= 6$, the ſecond Term $n \times \frac{n-3}{2}$ would be $= 9$, the third Term $n \times \frac{n-4}{2} \times \frac{n-5}{3}$ would be $= 2$. The reſt of the Terms vaniſhing in this Caſe. Wherefore if K, L, M are the three laſt Coefficients, the Coefficient of the following Term will be $6M - 9L + 2K$.

Fourthly, The Coefficient of any Term of theſe Series may be found independently from any relation they may have to the preceding: in order to which, it is to be obſerved that each Term of theſe Series is proportional to the Probability of the Play's ending in a certain number of Games preciſely: thus in the Series which expreſſes the Expectation of *R*, when each Gameſter is ſuppoſed to have 6 Pieces; *viz.*

$$\frac{a^6+b^6}{\overline{a+b}|^6} \times \overline{1 + \frac{6ab}{\overline{a+b}|^2} + \frac{27aabb}{\overline{a+b}|^4} + \frac{110a^3b^3}{\overline{a+b}|^6} + \frac{429a^4b^4}{\overline{a+b}|^8}}$$

the laſt Term being multiplied by the common Multiplicator $\frac{a^6+b^6}{\overline{a+b}|^6}$ ſet down before the Series, the Product $\frac{429a^4b^4 \times \overline{a^6+b^6}}{\overline{a+b}|^{14}}$ will denote the Probability of the Play's ending in 14 Games preciſely. Wherefore if that Term were deſired which expreſſes the Probability of the Play's ending in 20 Games preciſely, or in any number of Games denoted by $n+d$, I ſay that the Coefficient of that Term will be

$\frac{n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4} \times \frac{n+d-4}{5}$, &c. continued to ſo many Terms as there are Units in $\frac{1}{2}d$.

$-\frac{3n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4} \times \frac{n+d-4}{5}$, &c. continued to ſo many Terms as there are Units in $\frac{1}{2}d-n$.

$+\frac{5n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4} \times \frac{n+d-4}{5}$, &c. continued to ſo many Terms as there are Units in $\frac{1}{2}d-2n$.

$-\frac{7n}{1} \times \frac{n+d-1}{2} \times \frac{n+d-2}{3} \times \frac{n+d-3}{4} \times \frac{n+d-4}{5}$, &c. continued to ſo many Terms as there are Units in $\frac{1}{2}d-3n$.

&c.

Let now $n+d$ be ſuppoſed $=20$, n being already ſuppoſed $=6$, then the Coefficient demanded will be found from the general Rule to be

$$\begin{aligned} &\frac{6}{1} \times \frac{19}{2} \times \frac{18}{3} \times \frac{17}{4} \times \frac{16}{5} \times \frac{15}{6} \times \frac{14}{7} &&= 23256 \\ -&\frac{18}{1} &&= -18 \end{aligned}$$

Wherefore the Coefficient demanded will be $23256 - 18 = 23238$, and then the Term itſelf to which this Coefficient does belong, will be $\frac{23238a^7b^7}{\overline{a+b}|^{14}}$, and conſequently the Probability of the Play's ending in 20 Games preciſely will be $\frac{a^6+b^6}{\overline{a+b}|^6} \times \frac{23238a^7b^7}{\overline{a+b}|^{14}}$.

But ſome things are to be obſerved about this formation of the Coefficients, which are,

Firſt, that whenever it happens that $\frac{1}{2}d$, or $\frac{1}{2}d - n$, or $\frac{1}{2}d - 2n$, or $\frac{1}{2}d - 3n$, &c. expreſſing reſpectively the number of Multiplicators to be taken in each Line, are $= 0$, then 1 ought to be taken to ſupply that Line.

Secondly, That whenever it happens that thoſe quantities $\frac{1}{2}d$, or $\frac{1}{2}d - n$, or $\frac{1}{2}d - 2n$, or $\frac{1}{2}d - 3n$, &c. are leſs than nothing, otherwiſe that they are negative, then the Line to which they belong, as well as all the following, ought to be cancelled.

PROBLEM LX.

Suppoſing A *and* B *to play together till ſuch time as four Stakes are won or loſt on either ſide; what muſt be their proportion of Skill, otherwiſe what muſt be their proportion of Chances for winning any one Game aſſigned, to make it as probable that the Play will be ended in four Games as not?*

SOLUTION.

The Probability of the Play's ending in four Games is by the preceding Problem $\frac{a^4+b^4}{\overline{a+b}|^4} \times 1$: now becauſe, by Hypotheſis, it is to be an equal Chance whether the Play ends or ends not in four Games; let this Expreſſion of the Probability be made $= \frac{1}{2}$, then we ſhall have the Equation $\frac{a^4+b^4}{\overline{a+b}|^4} = \frac{1}{2}$: which, making $b, a :: 1, z$, is reduced to $\frac{z^4+1}{\overline{z+1}|^4} = \frac{1}{2}$, or $z^4 - 4z^3 - 6zz - 4z + 1 = 0$. Let $12zz$ be added on both ſides of the Equation, then will $z^4 - 4z^3 + 6zz - 4z + 1$ be $= 12zz$, and extracting the

Square-root on both ſides, it will be reduced to this *quadratic* Equation, $zz - 2z + 1 = z\sqrt{12}$, of which the two Roots are $z = 5.274$ and $z = \frac{1}{5.274}$. Wherefore whether the Skill of *A* be to that of *B*, as 5.274 to 1, or as 1 to 5.274, there will be an Equality of Chance for the Play to be ended or not ended in four Games.

PROBLEM LXI.

Suppoſing that A *and* B *play till ſuch time as four Stakes are won or loſt: What muſt be their proportion of Skill to make it a Wager of three to one, that the Play will be ended in four Games?*

SOLUTION.

The Probability of the Play's ending in four Games ariſing from the number of Games 4, from the number of Stakes 4, and from the proportion of Skill, *viz.* of *a* to *b*, is $\frac{a^4+b^4}{\overline{a+b}|^4}$; the ſame Probability ariſing from the Odds of three to one, is $\frac{3}{4}$: Wherefore $\frac{a^4+b^4}{\overline{a+b}|^4} = \frac{3}{4}$, and ſuppoſing $b, a :: 1, z$, that Equation will be changed into $\frac{z^4+1}{\overline{z+1}|^4} = \frac{3}{4}$ or $z^4 - 12z^3 + 38zz - 12z + 1 = 56zz$, and extracting the Square Root on both ſides, $zz - 6z + 1 = z\sqrt{56}$, the Roots of which Equation will be found to be 13.407 and $\frac{1}{13.407}$: Wherefore if the Skill of either be to that of the other as 13.407 to 1, 'tis a Wager of three to one, that the Play will be ended in 4 Games.

PROBLEM LXII.

Suppoſing that A *and* B *play till ſuch time as four Stakes are won or loſt; What muſt be their proportion of Skill to make it an equal Wager that the Play will be ended in ſix Games?*

SOLUTION.

The Probability of the Play's ending in ſix Games, ariſing from the given number of Games 6, from the number of Stakes 4, and from the proportion of Skill a to b, is $\frac{a^4+b^4}{\overline{a+b}|^4} \times \frac{\overline{1+4ab}}{\overline{a+b}|^2}$; the ſame Probability ariſing from an equality of Chance, is $= \frac{1}{2}$, from whence reſults the Equation $\frac{a^4+b^4}{\overline{a+b}|^4} \times \frac{\overline{1+4ab}}{\overline{a+b}|^2} = \frac{1}{2}$, which making $b, a :: 1, z$ muſt be changed into the following $z^6 + 6z^5 - 13z^4 - 20z^3 - 13zz + 6z + 1 = 0$.

In this Equation, the Coefficients of the Terms equally diſtant from the Extreams, being the ſame, let it be ſuppoſed that the Equation is generated from the Multiplication of two other Equations of the ſame nature, *viz.* $zz - yz + 1 = 0$, and $z^4 + pz^3 + qzz + pz + 1 = 0$. Now the Equation reſulting from the Multiplication of thoſe two will be

$$\begin{array}{llllll} z^6 & -yz^5 & +1z^4 & +2pz^3 & +pz & +1 = 0. \\ & +pz^5 & -pyz^4 & -qyz^3 & -yz & \\ & & +qz^4 & & & \end{array}$$

which being compared with the firſt Equation, we ſhall have $p - y = 6$, $1 - py + q = -13$, $2p - qy = -20$, from whence will be deduced a new Equation, *viz.* $y^3 + 6yy - 16y - 32 = 0$, of which one of the Roots will be 2.9644, and this being ſubſtituted in the Equation $zz - yz + 1 = 0$, we ſhall at laſt come to the Equation $zz - 2.9644z + 1 = 0$, of which the two Roots will be 2.576 and $\frac{1}{2.576}$; it follows therefore that if the Skill of either Gameſter be to that of the other as 2.576 to 1, there will be an equal Chance for four Stakes to be loſt or not to be loſt, in ſix Games.

COROLLARY

If the Coefficients of the extream Terms of an Equation, and likewiſe the Coefficients of the other Terms equally diſtant from the Extreams be the ſame, that Equation will be reducible to another, in which the Dimenſions of the higheſt Term will not exceed half the Dimenſions of the higheſt Term in the former.

PROBLEM LXIII.

Suppoſing A *and* B *whoſe proportion of Skill is as* a *to* b, *to play together till ſuch time as* A *either wins a certain number* q *of Stakes, or* B *ſome other number* p *of them: what is the Probability that the Play will not be ended in a given number of Games* (n)?

SOLU-

SOLUTION.

Multiply the Binomial $a+b$ ſo many times by it ſelf as there are Units in $n-1$, always obſerving after every Multiplication to reject thoſe Terms in which the Dimenſions of the Quantity a exceed the Dimenſions of the Quantity b, by q; as alſo thoſe Terms in which the Dimenſions of the Quantity b exceed the Dimenſions of the Quantity a, by p; then ſhall the laſt Product be the Numerator of a Fraction expreſſing the Probability required, of which Fraction the Denominator muſt be the Binomial $a+b$ raiſed to that Power which is denoted by n.

EXAMPLE.

Let p be $=3$, $q=2$, and let the given number of Games be $=7$. Let now the following Operation be made according to the foregoing Directions.

$$\begin{array}{l}
a+b \\
a+b \\
\hline
aa| + 2ab + bb \\
\qquad a+b \\
\hline
\quad 2aab + 3abb| + b^3 \\
\qquad a+b \\
\hline
\quad 2a^3b| + 5aabb + 3ab^3 \\
\qquad\qquad a+b \\
\hline
\qquad\quad 5a^3bb + 8aab^3| + 3ab^4 \\
\qquad\qquad a+b \\
\hline
\qquad\quad 5a^4bb| + 13a^3b^3 + 8aab^4 \\
\qquad\qquad\qquad a+b \\
\hline
\qquad\qquad 13a^4b^3 + 21a^3b^4| + 8ab^5 \\
\hline
\end{array}$$

From this Operation we may conclude, that the Probability of the Play's not ending in 7 Games is equal to $\frac{13a^4b^3+21a^3b^4}{\overline{a+b}^{\,7}}$. Now if an equality of Skill be ſuppoſed between A and B, the Expreſſion of this Probability will be reduced to $\frac{13+21}{128}$ or $\frac{17}{64}$: Wherefore the Probability of the Play's ending in 7 Games will be $\frac{47}{64}$; from which it follows that it is 47 to 17 that, in ſeven Games, either A wins two Stakes of B, or B wins three Stakes of A.

PROBLEM LXIV.

The ſame things being ſuppoſed as in the preceding Problem, to find the Probability of the Play's ending in a given number of Games.

SOLUTION.

Firſt, If the Probability of the Play's not ending in the given number of Games, which we may obtain from the preceding Problem, be ſubtracted from Unity, there will remain the Probability of its ending in the ſame number of Games.

Secondly, This Probability may be expreſſed by means of the Terms rejected in the Operation belonging to the preceding Problem: Thus if the number of Stakes be 3 and 2, the Probability of the Play's ending in 7 Games may be expreſſed as follows.

$$\frac{aa}{(a+b)^2}\times\overline{1+\frac{2ab}{(a+b)^2}+\frac{5aabb}{(a+b)^4}}$$
$$\frac{b^3}{(a+b)^3}\times\overline{1+\frac{3ab}{(a+b)^2}+\frac{8aabb}{(a+b)^4}}$$

Suppoſing both a and b equal to Unity, the Sum of the firſt Series will be $=\frac{29}{64}$, and the Sum of the ſecond will be $\frac{18}{64}$; which two Sums being added together, the aggregate $\frac{47}{64}$ expreſſes the Probability that, in ſeven Games, either A ſhall win two Stakes of B, or B three Stakes of A.

Thirdly, The Probability of the Play's ending in a certain number of Games is always compoſed of a double Series, when the Stakes are unequal: which double Series is reduced to a ſingle one, in the Caſe of an Equality of Stakes

The firſt Series always expreſſes the Probability there is that A, in a given number of Games, or ſooner, may win of B the number q of Stakes, excluding the Probability there is that B before that time may have been in a circumſtance of winning the number p of Stakes; both which Probabilities are not inconſiſtent together: for A, in fifteen Games for Inſtance or ſooner, may win two Stakes of B, though B before that time may have been in a circumſtance of winning three Stakes of A.

The ſecond Series always expreſſes the Probability there is that B, in that given number of Games, may win of A a certain number p of Stakes, excluding the Probability there is that *A*, before that time, may win of *B* the number q of Stakes.

The firſt Terms of each Series may be repreſented reſpectively by the following Terms.

$$\frac{a^q}{\overline{a+b}|^q} \times \overline{1 + \frac{qab}{\overline{a+b}|^2} + \frac{q \cdot \overline{q+3} \cdot aabb}{1 \cdot 2 \cdot \overline{a+b}|^4} + \frac{q \cdot \overline{q+4} \cdot \overline{q+5} \cdot a^3b^3}{1 \cdot 2 \cdot 3 \cdot \overline{a+b}|^6} + \frac{q \cdot \overline{q+5} \cdot \overline{q+6} \cdot \overline{q+7} \cdot a^4b^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \overline{a+b}|^8}}, \text{ \&c.}$$

$$\frac{a^p}{\overline{a+b}|^p} \times \overline{1 + \frac{pab}{\overline{a+b}|^2} + \frac{p \cdot \overline{p+3} \cdot aabb}{1 \cdot 2 \cdot \overline{a+b}|^4} + \frac{p \cdot \overline{p+4} \cdot \overline{p+5} \cdot a^3b^3}{1 \cdot 2 \cdot 3 \cdot \overline{a+b}|^6} + \frac{p \cdot \overline{p+5} \cdot \overline{p+6} \cdot \overline{p+7} \cdot a^4b^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \overline{a+b}|^4}}, \text{ \&c.}$$

Each of theſe Series continuing in that regularity till ſuch time as there be a number p of Terms taken in the firſt, and a number q of Terms taken in the ſecond; after which the Law of the continuation breaks off.

Now in order to find any of the Terms following in either of theſe Series, proceed thus: let $p + q - 2$ be called l; let the Coefficient of the Term deſired be T; let alſo the Coefficients of the preceding Terms taken in an inverted order, be S, R, Q, P, &c. then will T be equal to $lS - \frac{l-1}{1} \times \frac{l-2}{2}R + \frac{l-2}{1} \times \frac{l-3}{2} \times \frac{l-4}{3}Q - \frac{l-3}{1} \times \frac{l-4}{2} \times \frac{l-5}{3} \times \frac{l-6}{4}P$, &c. Thus if p be $= 3$ and $q = 2$. then l will be $3 + 2 - 2 = 3$, wherefore $lS - \frac{l-1}{1} \times \frac{l-2}{2}R$ would in this Caſe be equal to $3S - R$, which ſhews that the Coefficient of any Term deſired would be three times the laſt, *minus* once the laſt but one.

To apply this, let it be required to find what Probability there is that in fifteen Games or ſooner, either *A* ſhall win two Stakes of *B*, or *B* three Stakes of *A*; or which is all one, to find what Probability there is that the Play ſhall end in fifteen Games at fartheſt; *A* and *B* reſolving to play till ſuch time as *A* either wins two Stakes or *B* three.

Let 2 and 3, in the two foregoing Series, be ſubſtituted reſpectively in the room of q and p, the three firſt Terms of the firſt Series will be, ſetting aſide the common Multiplicator, $1 + \frac{2ab}{\overline{a+b}|^2} + \frac{5aabb}{\overline{a+b}|^4}$: likewiſe the two firſt Terms of the ſecond will be $1 + \frac{3ab}{\overline{a+b}|^2}$. Now becauſe the Coefficient of any Term deſired in each Series is reſpectively three times the laſt, *minus* once the laſt but one, it follows that the next Coefficient in the firſt Series will be found to be 13, and by the ſame Rule the next to it 34, and ſo on In the ſame manner, the next Coefficient in the ſecond Series will be found to be 8, and the next to it 21, and ſo on. Wherefore reſtoring the common Multiplicators the two Series will be

$$\frac{a^2}{\overline{a+b}\vert^{2}} \times \overline{1 + \frac{2ab}{\overline{a+b}\vert^{2}} + \frac{5aabb}{\overline{a+b}\vert^{4}} + \frac{13a^3b^3}{\overline{a+b}\vert^{6}} + \frac{34a^4b^4}{\overline{a+b}\vert^{8}} + \frac{89a^5b^5}{\overline{a+b}\vert^{10}} + \frac{233a^6b^6}{\overline{a+b}\vert^{12}}}\,.$$

$$\frac{b^3}{\overline{a+b}\vert^{3}} \times \overline{1 + \frac{3ab}{\overline{a+b}\vert^{2}} + \frac{8aabb}{\overline{a+b}\vert^{4}} + \frac{21a^3b^3}{\overline{a+b}\vert^{6}} + \frac{55a^4b^4}{\overline{a+b}\vert^{8}} + \frac{144a^5b^5}{\overline{a+b}\vert^{10}} + \frac{377a^6b^6}{\overline{a+b}\vert^{12}}}\,.$$

If we ſuppoſe an equality of Skill between *A* and *B*, the Sum of the firſt Series will be $\frac{18778}{32768}$, the Sum of the ſecond will be $\frac{12393}{32768}$, and the Aggregate of thoſe two Sums will be $\frac{31171}{32768}$, which will expreſs the Probability of the Play's ending in fifteen Games or ſooner. This laſt Fraction being ſubtracted from Unity, there will remain $\frac{1597}{32768}$, which expreſſes the Probability of the Play's continuing beyond fifteen Games: Wherefore 'tis 31171 to 1597, or 39 to 2 nearly that one of the two equal Gameſters that ſhall be pitched upon, ſhall in fifteen Games at fartheſt, either win two Stakes of his Adverſary, or loſe three to him.

N. B. The Index of the Denominator in the laſt Term of each Series, and the Index of the common Multiplicator prefixed to it being added together, muſt either equal the number of Games given, or be leſs than it by Unity. Thus in the firſt Series, the Index 12 of the Denominator of the laſt Term, and the Index 2 of the common Multiplicator being added together, the Sum is 14, which is leſs by Unity than the number of Games given. So likewiſe in the ſecond Series, the Index 12 of the Denominator of the laſt Term, and the Index 3 of the common Multiplicator being added together, the Sum is 15, which preciſely equals the number of Games given.

It is carefully to be obſerved that thoſe two Series taken together expreſs the Expectation of one and the ſame perſon, and not of two different perſons; that is properly of a Spectator, who lays a wager that the Play will be ended in a given number of Games. Yet in one Caſe, they may expreſs the Expectations of two different perſons: for Inſtance, of the Gameſters themſelves, provided that both Series be continued infinitely; for in that Caſe, the firſt Series infinitely continued will expreſs the Probability that the Gameſter *A* may ſooner win two Stakes of *B*, than that he may loſe three to him: likewiſe the ſecond Series infinitely continued will expreſs the Probability that the Gameſter *B* may ſooner win three Stakes of *A*, than loſe two to him. And it will be found, (when I come to treat of the Method of ſumming up this ſort of Series, whoſe Terms have a perpetual recurrency of relation to a fixed number of preceding Terms) that the firſt Series infinitely continued is to the ſecond infinitely continued, in the proportion of $aa \times \overline{aa + ab + bb}$ to $b^3 \times \overline{a + b}$; that is in the Caſe of an Equality of Skill as 3 to 2, which is conformable to what I have ſaid in the IXth Problem.

Fourthly, Any Term of theſe Series may be found independently from any of the preceding: for if a Wager be laid that *A* ſhall either win a certain number of Stakes denominated by q, or that *B* ſhall win a certain number of them denominated by p, and that the number of Games be expreſſed by $q + d$; then I ſay that the Coefficient of any Term in the firſt Series anſwering to that number of Games will be

$+ \frac{q}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4}$, &c. continued to ſo many Multiplicators as there are Units in $\frac{1}{2}d$.

$- \frac{q+2p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4}$, &c. continued to ſo many Terms as there are Units in $\frac{1}{2}d - p$.

$+ \frac{3q+2p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4}$, &c. continued to ſo many Terms as there are Units in $\frac{1}{2}d - p - q$.

$- \frac{3q+4p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4}$, &c. continued to ſo many Terms as there are Units in $\frac{1}{2}d - 2p - q$.

$+ \frac{5q+4p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4}$, &c. continued to ſo many Terms as there are Units in $\frac{1}{2}d - 2p - 2q$.

$- \frac{5q+6p}{1} \times \frac{q+d-1}{2} \times \frac{q+d-2}{3} \times \frac{q+d-3}{4}$, &c. continued to ſo many Terms as there are Units in $\frac{1}{2}d - 3p - 2q$.

$+ \frac{7q+6p}{1} \times \frac{q+d-1}{2} \times \frac{q+d+2}{3} \times \frac{q+d-3}{4}$, &c. continued to ſo many Terms as there are Units in $\frac{1}{2}d-3p-3q$.
And ſo on.

And the ſame Law will hold for the other Series, calling $p+\delta$ the number of Games given, and changing q into p, and p into q, as alſo d into δ, ſtill remembring that when d is an odd number, $d-1$ ought to be taken in the room of it, and the like for δ.

And the ſame obſervation muſt be made here as was made at the end of the LIXth Problem, *viz.* that if $\frac{1}{2}d$, or $\frac{1}{2}d-p$, or $\frac{1}{2}d-p-q$, or $\frac{1}{2}d-2p-q$, or $\frac{1}{2}d-2p-2q$, &c. expreſſing reſpectively the number of Multiplicators to be taken in each Line, are $=0$, then 1 ought to be taken for that Line, and alſo, that if $\frac{1}{2}d$, or $\frac{1}{2}d-p$, or $\frac{1}{2}d-p-q$, &c. are leſs than nothing, otherwiſe negative, then the Line to which they belong as well as all the following ought to be cancelled.

PROBLEM LXV.

If A *and* B, *whoſe proportion of ſkill is ſuppoſed as* a *to* b, *play together: What is the Probability that one of them, ſuppoſe* A, *may in a number of Games not exceeding a number given, win of* B *a certain number of Stakes? leaving it wholly indifferent whether* B, *before the expiration of thoſe Games, may or may not have been in a circumſtance of winning the ſame, or any other number of Stakes of* A.

SOLUTION.

Suppoſing n to be the number of Stakes which A is to win of B, and $n+d$ the number of Games; let $a+b$ be raiſed to the Power whoſe Index is $n+d$; then if d be an odd number, take ſo many Terms of that Power as there are Units in $\frac{d+1}{2}$; take alſo ſo many of the Terms next following as have been taken already, but prefix to them in an inverted order, the Coefficients of the preceding Terms. But if d be an even-number, take ſo many Terms of the ſaid Power as there are Units in $\frac{1}{2}d+1$; then take as many of the Terms next following as there are Units in $\frac{1}{2}d$, and prefix to them in an inverted order the Coefficients of the preceding Terms, omitting the laſt of them; and thoſe Terms taken all together will compoſe the Numerator of a Fraction expreſſing the Probability required, the Denominator of which Fraction ought to be $\overline{a+b}|^{n+d}$.

EXAMPLE I.

Suppoſing the number of Stakes, which *A* is to win, to be *Three*, and the given number of Games to be *Ten*; let $a+b$ be raiſed to the tenth power, *viz.* $a^{10}+10a^9b+45a^8bb+120a^7b^3+210a^6b^4+252a^5b^5+210a^4b^6+120a^3b^7+45aab^8+10ab^9+b^{10}$. Then by reaſon that $n=3$, and $n+d=10$, it follows that d is $=7$, and $\frac{d+1}{2}=4$. Wherefore let the Four firſt Terms of the ſaid Power be taken, *viz.* $a^{10}+10a^9b+45a^8bb+120a^7b^3$, and let the four Terms next following be taken likewiſe without regard to their Coefficients, then prefix to them in an inverted order, the Coefficients of the preceding Terms: thus the four Terms following with their new Coefficients will be $120a^6b^4+45a^5b^5+10a^4b^6+1a^3b^7$. Then the Probability which *A* has of winning three Stakes of *B* in ten Games or ſooner, will be expreſſed by the following Fraction

$$\frac{a^{10}+10a^9b+45a^8bb+120a^7b^3+120a^6b^4+45a^5b^5+10a^4b^6+a^3b^7}{\overline{a+b}|^{10}}$$

which in the Caſe of an Equality of Skill between *A* and *B* will be reduced to $\frac{352}{1024}$ or $\frac{11}{32}$.

EXAMPLE II.

Suppoſing the number of Stakes which *A* has to win to be *Four*, and the given number of Games to be *Ten*; let $a+b$ be raiſed to the tenth Power, and by reaſon that n is $=4$, and $n+d=10$, it follows that d is $=6$, and $\frac{1}{2}d+1=4$; wherefore let the four firſt Terms of the ſaid Power be taken, *viz.* $a^{10}+10a^9b+45a^8bb+120a^7b^3$; take alſo three of the Terms following, but prefix to them, in an inverted order, the Coefficients of the Terms already taken, omitting the laſt of them; hence the three Terms following with their new Coefficients will be $45a^6b^4+10a^5b^5+1a^4b^6$. Then the Probability which *A* has of winning four Stakes of *B* in ten Games, or ſooner, will be expreſſed by the following Fraction

$$\frac{a^{10}+10a^9b+45a^8bb+120a^7b^3+45a^6b^4+10a^5b^5+1a^4b^6}{\overline{a+b}|^{10}}$$

which in the Caſe of an Equality of Skill between *A* and *B* will be reduced to $\frac{232}{1024}$ or $\frac{29}{128}$.

Another SOLUTION.

Suppoſing as before that n be the number of Stakes which *A* is to win, and that the number of Games be $n+d$, the Probability which *A* has of winning will be expreſſed by the following Series

$$\frac{a^n}{\overline{a+b}|^n} \times \overline{1+\frac{nab}{\overline{a+b}|^2}+\frac{n\,.\,\overline{n+3}\,.\,aabb}{1\,.\,2\,.\,\overline{a+b}|^4}+\frac{n\,.\,\overline{n+4}\,.\,\overline{n+5}\,.\,a^3b^3}{1\,.\,2\,.\,3\,.\,\overline{a+b}|^6}+\frac{n\,.\,\overline{n+5}\,.\,\overline{n+6}\,.\,\overline{n+7}\,.\,a^4b^4}{1\,.\,2\,.\,3\,.\,4\,.\,\overline{a+b}|^8}}$$

, &c. which Series ought to be continued to ſo many Terms as there are Units in $\frac{1}{2}d+1$; always obſerving to ſubſtitute $d-1$ in the room of d in Caſe d be an odd number, or which is the ſame thing, taking ſo many Terms as there are Units in $\frac{d+1}{2}$.

Now ſuppoſing, as in the firſt Example of the preceding Solution, that Three is the number of Stakes, and Ten the given number of Games, and alſo that there is an equality of Skill between *A* and *B*, the foregoing Series will become $\frac{1}{8} \times \overline{1+\frac{3}{4}+\frac{9}{16}+\frac{28}{64}} = \frac{11}{32}$, as before.

REMARK.

In the firſt attempt that I had ever made towards ſolving the general Problem of the Duration of Play, which was in the Year 1708, I began with the Solution of this LXV^th^ Problem, well knowing that it might be a Foundation for what I farther wanted, ſince which time, by a due repetition of it, I ſolved the main Problem: but as I found afterwards a nearer way to it, I barely publiſhed in my firſt Eſſay on thoſe matters, what ſeemed to me moſt ſimple and elegant, ſtill preſerving this Problem by me in order to be publiſhed when I ſhould think it proper. Now in the year 1713 Mr. *de Monmort* printed a Solution of it in a Book by him publiſhed upon Chance, in which was alſo inſerted a Solution of the ſame by Mr. *Nicolas Bernoulli*; and as thoſe two Solutions ſeemed to me, at firſt ſight, to have ſome affinity with what I had found before, I conſidered them with very great attention; but the Solution of Mr. *Nicolas Bernoulli* being very much crouded with Symbols, and the verbal Explication of them too ſcanty, I own I did not underſtand it thoroughly, which obliged me to conſider Mr. *de Monmort*'s Solution with very great attention: I found indeed that he was very plain, but to my great ſurprize I found him very erroneous; ſtill in my Doctrine of Chances I printed that Solution, but rectified and aſcribed it to Mr. *de Monmort*, without the leaſt intimation of any alterations made by me; but as I had no thanks for ſo doing, I reſume my right, and now print it as my own: but to come to the Solution.

Let it be propoſed to find the number of Chances there are for *A* to win two Stakes of *B*, or for *B* to win three Stakes of *A*, in fifteen Games.

The number of Chances required is expreſſed by two Branches of Series; all the Series of the firſt Branch taken together expreſs the number of Chances there are for *A* to win two Stakes of *B*, excluſive of the number of Chances there are for *B* before that time, to win three Stakes of *A*. All the Series of the ſecond Branch taken together expreſs the number of Chances there are for *B* to win three Stakes of *A*, excluſive of the number of Chances there are for *A* before that time to win two Stakes of *B*.

Firſt Branch of Series.

$$\begin{array}{cccccccccccccc}
a^{15} & a^{14}b & a^{13}b^{2} & a^{12}b^{3} & a^{11}b^{4} & a^{10}b^{5} & a^{9}b^{6} & a^{8}b^{7} & a^{7}b^{8} & a^{6}b^{9} & a^{5}b^{10} & a^{4}b^{11} & a^{3}b^{12} & a^{2}b^{13} \\
1 & +15 & +105 & +455 & +1365 & +3003 & +5005 & +5005 & +3003 & +1365 & +455 & +105 & +15 & +1 \\
 & & & -1 & -15 & -105 & -455 & -455 & -105 & -15 & -1 & & & \\
 & & & & & +1 & +15 & +15 & +1 & & & & &
\end{array}$$

Second Branch of Series.

$$\begin{array}{ccccccccccccc}
b^{15} & b^{14}a & b^{13}a^{2} & b^{11}a^{4} & b^{10}a^{5} & b^{9}a^{6} & b^{8}a^{7} & b^{7}a^{8} & b^{6}a^{9} & b^{5}a^{10} & b^{4}a^{11} & b^{3}a^{12} & b^{2}a^{13} \\
1 & +15 & +105 & +455 & +1365 & +3003 & +5005 & +3003 & +1365 & +455 & +105 & +15 & +1 \\
 & & -1 & -15 & -105 & -455 & -1365 & -455 & -105 & -15 & -1 & & \\
 & & & & & +1 & +15 & +1 & & & & &
\end{array}$$

The literal Quantities which are commonly annexed to the numerical ones, are here written on the top of them; which is done, to the end that each Series being contained in one Line, the dependency they have upon one another, may thereby be made more conſpicuous.

The firſt Series of the firſt Branch expreſſes the number of Chances there are for *A* to win two Stakes of *B*, including the number of Chances there are for *B* before, or at the Expiration of the fifteen Games, to be in a Circumſtance of winning three Stakes of *A*; which number of Chances may be deduced from the LXVth Problem.

The ſecond Series of the firſt Branch is a part of the firſt, and expreſſes the number of Chances there are for *B* to win three Stakes of *A*, out of the number of Chances there are for *A*, in the firſt Series to win two Stakes of *B*. It is to be obſerved about this Series, *Firſt*, that the Chances of *B* expreſſed by it are not reſtrained to happen in any order, that is, either before or after *A* has won two Stakes of *B*. *Secondly*, that the literal products belonging to it are the ſame with thoſe of the correſponding Terms of the firſt Series. *Thirdly*, that it begins and ends at an Interval from the firſt and laſt Terms of the firſt Series equal to the number of Stakes which *B* is to win. *Fourthly*, that the numbers belonging to it are the numbers of the firſt Series repeated in order, and continued to one half of its Terms; after which thoſe numbers return in an inverted order to the end of that Series: which is to be underſtood in caſe the number of its Terms ſhould happen to be even; for if it ſhould happen to be odd, then that order is to be continued to the greateſt half, after which the return is made by omitting the laſt number. *Fifthly*, that all the Terms of it are affected with the ſign *minus*.

The Third Series is part of the ſecond, and expreſſes the number of Chances there are for *A* to win two Stakes of *B*, out of the number of Chances there are in the ſecond Series for *B* to win three Stakes of *A*; with this difference, that it begins and ends at an Interval from the firſt and laſt Terms of the ſecond Series, equal to the number of Stakes which *A* is to win; and that the Terms of it are all poſitive.

It is to be obſerved, that let the number of thoſe Series be what it will, the Interval between the beginning of the firſt and the beginning of the ſecond, is to be equal to the number of Stakes which *B* is to win; and that the Interval between the beginning of the ſecond and the beginning of the third, is to be equal to the number of Stakes which *A* is to win; and that theſe Intervals recur alternately in the ſame order. It is to be obſerved likewiſe that all theſe Series are alternately poſitive and negative.

All the Obſervations made upon the firſt Branch of Series belonging alſo to the ſecond, it would be needleſs to ſay any thing more of them.

Now the Sum of all the Series of the firſt Branch, being added to the Sum of all the Series of the ſecond, the Aggregate of theſe Sums will be the Numerator of a Fraction expreſſing the Probability of the Play's terminating in the given number of Games; of which the Denominator is the Binomial $a+b$ raiſed to a Power whoſe Index is equal to that number of Games. Thus ſuppoſing that in the Caſe of this Problem both a and b are equal to Unity, the Sum of the Series in the firſt Branch will be 18778, the Sum of the Series in the ſecond will be 12393, and the Aggregate of both 31171; and the Fifteenth Power of 2 being 32768, it follows that the Probability of the Play's terminating in Fifteen Games will be $\frac{31171}{32768}$, which being ſubtracted from Unity, the remainder will be $\frac{1597}{32768}$: From whence we may conclude that it is a Wager of 31171 to 1597, that either A in Fifteen Games ſhall win two Stakes of B, or B win three Stakes of A: which is conformable to what was found in the LXIV^th Problem.

PROBLEM LXVI.

To find what Probability there is that in a given number of Games A *may be winner of a certain number* q *of Stakes, and at ſome other time* B *may likewiſe be winner of the number* p *of Stakes, ſo that both circumſtances may happen.*

SOLUTION.

Find by our LXV^th Problem the Probability which A has of winning, without any limitation, the number q of Stakes: find alſo by the LXIII^d Problem the Probability which A has of winning that number of Stakes before B may happen to win the number p; then from the firſt Probability ſubtracting the ſecond, the remainder will expreſs the Probability there is that both A and B may be in a circumſtance of winning, but B before A. In the like manner, from the Probability which B has of winning without limitation, ſubtracting the Probability which he has of winning before A, the remainder will expreſs the Probability there is that both A and B may be in a circumſtance of winning, but A before B: wherefore adding theſe two remainders together, their Sum will expreſs the Probability required.

Thus if it were required to find what Probability there is, that in Ten Games *A* may win Two Stakes of *B*, and that at ſome other time *B* may win Three:

The firſt Series will be found to be

$$\frac{aa}{\overline{a+b}|^2}\times\overline{1+\frac{2ab}{\overline{a+b}|^2}+\frac{5aabb}{\overline{a+b}|^4}+\frac{14a^3b^3}{\overline{a+b}|^6}+\frac{42a^4b^4}{\overline{a+b}|^8}}$$

The ſecond Series will be

$$\frac{aa}{\overline{a+b}|^2}\times\overline{1+\frac{2ab}{\overline{a+b}|^2}+\frac{5aabb}{\overline{a+b}|^4}+\frac{13a^3b^3}{\overline{a+b}|^6}+\frac{34a^4b^4}{\overline{a+b}|^8}}$$

The difference of theſe Series being $\frac{aa}{\overline{a+b}|^2}\times\overline{\frac{a^3b^3}{\overline{a+b}|^6}+\frac{8a^4b^4}{\overline{a+b}|^8}}$ expreſſes the firſt part of the Probability required, which in the Caſe of an equality of Skill between the Gameſters would be reduced to $\frac{3}{256}$.

The third Series is as follows,

$$\frac{b^3}{\overline{a+b}|^3}\times\overline{1+\frac{3ab}{\overline{a+b}|^2}+\frac{9aabb}{\overline{a+b}|^4}+\frac{28a^3b^3}{\overline{a+b}|^6}}$$

The fourth Series is

$$\frac{b^3}{\overline{a+b}|^3}\times\overline{1+\frac{3ab}{\overline{a+b}|^2}+\frac{8aabb}{\overline{a+b}|^4}+\frac{21a^3b^3}{\overline{a+b}|^6}}$$

The difference of theſe two Series being $\frac{b^3}{\overline{a+b}|^3}\times\overline{\frac{aabb}{\overline{a+b}|^4}+\frac{7a^3b^3}{\overline{a+b}|^6}}$ expreſſes the ſecond part of the Probability required, which in the Caſe of an equality of Skill would be reduced to $\frac{11}{512}$. Wherefore the Probability required would in this Caſe be $\frac{3}{256}+\frac{11}{512}=\frac{17}{512}$. Whence it follows, that it is a Wager of 495 to 17, or 29 to 1 very near, that in Ten Games *A* and *B* will not both be in a circumſtance of winning, *viz.* *A* the number *q* and *B* the number *p* of Stakes. But if by the conditions of the Problem, it were left indifferent whether *A* or *B* ſhould win the two Stakes or the three, then the Probability required would be increaſed, and become as follows; *viz.*

$$\frac{aa+bb}{\overline{a+b}|^2}\times\overline{\frac{a^3b^3}{\overline{a+b}|^6}+\frac{8a^4b^4}{\overline{a+b}|^8}}$$
$$+\frac{a^3+b^3}{\overline{a+b}|^3}\times\overline{\frac{aabb}{\overline{a+b}|^4}+\frac{7a^3b^3}{\overline{a+b}|^6}}$$

which, in the Caſe of an equality of Skill between the Gameſters, would be double to what it was before.

PROBLEM LXVII.

To find what Probability there is, that in a given number of Games A *may win the number* q *of Stakes; with this farther condition, that* B *during that whole number of Games may never have been winner of the number* p *of Stakes.*

SOLUTION.

From the Probability which *A* has of winning without any limitation the number *q* of Stakes, ſubtract the Probability there is that both *A* and *B* may be winners, *viz.* *A* of the number *q*, and *B* of the number *p* of Stakes, and there will remain the Probability required.

But if the conditions of the Problem were extended to this alternative, *viz.* that either *A* ſhould win the number *q* of Stakes, and *B* be excluded the winning of the number *p*; or that *B* ſhould win the number *p* of Stakes, and *A* be excluded the winning of the number *q*, the Probability that either the one or the other of theſe two Caſes may happen, will eaſily be deduced from what we have ſaid.

The Rules hitherto given for the Solution of Problems relating to the Duration of Play are eaſily practicable, if the number of Games given is but ſmall; but if that number is large, the work will be very tedious, and ſometimes ſwell to that degree as to be in ſome manner impracticable: to remedy which inconveniency, I ſhall here give an Extract of a paper by me produced before the Royal Society, wherein was contained a Method of ſolving very expeditiouſly the chief Problems relating to that matter, by the help of a Table of Sines, of which I had before given a hint in the firſt Edition of my *Doctrine of Chances*, pag. 149, and 150.

PROBLEM LXVIII.

To ſolve by a Method different from any of the preceding, the Problem LIX, *when* a *is to* b *in a ratio of Equality.*

SOLUTION.

Let n be the number of Games given, and p the number of Stakes; let Q repreſent 90 degrees of a Circle whoſe Radius is equal to Unity; let C, D, E, F, &c. be the Sines of the Arcs $\frac{Q}{p}$, $\frac{3Q}{p}$, $\frac{5Q}{p}$, $\frac{7Q}{p}$, &c. till the Quadrant be exhauſted; let alſo, c, d, e, f, &c. be the Co-ſines of thoſe Arcs: then if the difference between n and p be an even number, the Probability of the Play's not ending in the given number of Games will be repreſented by the Series

$$\frac{2}{p} \times \frac{c^{n+1}}{C} - \frac{d^{n+1}}{D} + \frac{e^{n+1}}{E} - \frac{f^{n+1}}{F}, \text{ \&c.}$$

of which Series very few Terms will be ſufficient for a very near approximation. But if the difference between n and p be odd, then the Probabity required will be $\frac{2}{p} \times \frac{c^{n}}{C} - \frac{d^{n}}{D} + \frac{e^{n}}{E} - \frac{f^{n}}{F}$ &c.

In working by Logarithms, you are perpetually to ſubtract, from the Logarithm of every Term, the Product of 10 into the number n, in caſe the number $n - p$ be even; but in caſe it be odd, you are to ſubtract the Product of 10 into $n - 1$, and if the Subtraction cannot be made without making the remainder negative, add 10, 20, or 30, &c. and make ſuch proper allowances for thoſe additions as thoſe who are converſant with Logarithms know how to make.

To apply this to ſome particular caſes, let it be required to find the Probability of Twelve Stakes being not loſt in 108 Games.

Here becauſe the difference between 108 and 12 is 96, I take the firſt form, thus

The Arcs $\frac{Q}{p}$, $\frac{3Q}{p}$, $\frac{5Q}{p}$, $\frac{7Q}{p}$, $\frac{9Q}{p}$, $\frac{11Q}{p}$, $\frac{13Q}{p}$, &c. being reſpectively $7^d - 30'$, $22^d - 30'$, $37^d - 30'$, $52^d - 30'$, $67^d - 30'$, $82^d - 30'$, $97^d - 30'$, &c. I take only the ſix firſt, as not exceeding 90^d.

Now the Logarithm of the Co-ſine of $7^d - 30'$ being 9.9962686, I multiply it by $n + 1$, that is in this Caſe by 109, and the product will be 1089.5932774, which is the Logarithm of the Numerator of the firſt Fraction $\frac{c^{n+1}}{C}$.

From that Logarithm, I ſubtract the Logarithm of the Sine of $7^d - 30'$ here repreſented by C, which being 9.1156977, the remainder will be 1080.4775797, out of which rejecting 1080 product of 10 by the given number of Games 108, and taking only 0.4775797 the number anſwering will be 3.00327, which being multiplied by the common Multiplicator $\frac{2}{p}$, that is in this Caſe by $\frac{2}{12}$ or $\frac{1}{6}$, the product will be 0.50053, which Term alone determines nearly the Probability required.

For if we intend to make a Correction by means of the ſecond Term $\frac{d^{n+1}}{D}$, we ſhall find the Logarithm of $\frac{d^{n+1}}{D}$ to be 1076.6692280 to which adding 10, and afterwards ſubtracting 1080, the remainder will be 6.6692280, to which anſwers 0.0004669, of which the 6th part is 0.0000778, which being almoſt nothing may be ſafely rejected. And whenever it happens that n is a large number in reſpect to p, the firſt Term alone of theſe Series will exceeding near determine the Probability required.

Let it now be required to find the Probability of 45 Stakes being not loſt on either ſide in 1519 Games.

The Arcs $\frac{Q}{p}$, $\frac{3Q}{p}$, $\frac{5Q}{p}$, &c. being reſpectively 2^d, 6^d, 10^d, &c. I take, 1°, the Logarithm of the Co-ſine of 2^d which is 9.9997354, which being multiplied by $n+1$, that is in this Caſe by 1520, the product will be 15199.5988080, out of which ſubtracting the Logarithm of the Sine of 2^d, *viz.* 8.5428192, the remainder will be 15191.0559888, out of which rejecting 15190, the number anſwering will be 11.3759, which being multiplied by $\frac{2}{p}$, that is, in this Caſe by $\frac{2}{45}$, the product will be .50559 which nearly determines the Probability required.

Now if we want a Correction by means of the ſecond Term, we ſhall find $\frac{d^{n+1}}{D} = .00002081$, which Term being ſo very inconſiderable may be entirely rejected, and much more all the following.

Conſidering therefore that when the Arc $\frac{Q}{p}$ is ſmall, the firſt Term alone is ſufficient for a near approximation, it will not be amiſs to inquire what muſt be the number of Games that ſhall make it an equal Probability of the Play's being ended in that number of Games; which to do, Suppoſe $\frac{c^{n+1}}{C} \times \frac{2}{p} = \frac{1}{2}$, hence $4c^{n+1} = Cp$, then ſuppoſing p a large number, whereby the number n muſt be ſtill much larger, we may barely take for our Equation $4c^{n} = pC$, then taking the Logarithms, we ſhall have Log. $4 + n$ Log. $c =$ Log. C $+$ Log. p, let the magnitude of the Arc $\frac{Q}{p}$ be ſuppoſed $= z$; now ſince the number p has been ſuppoſed very large, it follows that the Arc z muſt be very ſmall; wherefore the Sine of that Arc will alſo be nearly $= z$, and its Co-ſine $1 - \frac{1}{2}zz$ nearly, of which Co-ſine the Logarithm will be $- \frac{1}{2}zz$ nearly; we have therefore the Equation Log. $4 - \frac{1}{2}nzz =$ Log. $p +$ Log. z; let now the Magnitude of an Arc of 90′, to a Radius equal to Unity, be $= M$, hence we ſhall have $\frac{M}{p} = z$, and Log. $z =$ Log. M $-$ Log. p, wherefore the Equation will at laſt be changed into this, Log. 4 $- \frac{\frac{1}{2}nMM}{pp} =$ Log. M, and therefore $n = \frac{2 \log. 4 - 2 \log. M}{MM} \times pp$, but $\frac{2 \log. 4 - 2 \log. M}{MM} = 0.756$ nearly, and therefore $n = 0.756pp$.

N. B. The Logarithms here made uſe of are ſuppoſed to be Hyperbolic Logarithms, of which I hear a Table will ſoon be publiſhed.

Mr. *de Monmort* in the ſecond Edition of his Tract, *Des jeux de Hazard*, tells us that he found that if p denoted an odd number of Stakes to be won or loſt, making $\frac{p+1}{2} = ſ$, that then the Quantity $3ſſ - 3ſ + 1$ would denote a number of Games wherein there would be more than an equal Probability of the Play's being ended; but at the ſame time he owns, that he has not been able to find a Rule like it for an even number of Stakes.

Whereupon I ſhall obſerve, *firſt*, that his Expreſſion may be reduced to $\frac{3}{4}pp + \frac{1}{4}$. Which tho' near the Truth in ſmall numbers, yet is very defective in large ones, for it may be proved that the number of Games found by his Expreſſion, far from being above what is requiſite, is really below it. *Secondly*, that his Rule does not err more in an even number of Stakes than in an odd one; but that Rule being founded upon an induction gathered from the Solution of ſome of the ſimpleſt Caſes of this Problem, it is no wonder that he reſtrained it to the odd Caſes, he happening to be miſtaken in determining the number of Games requiſite to make it an even Wager that twelve Stakes would be won or loſt before or at the expiration of thoſe Games, which he finds by a very laborious calculation to have been 122; in which however he was afterwards rectified by Mr. *Nicolas Bernoulli*, who informed him that he had found by his own Calculation that the number of Games requiſite for that purpoſe was above 108, and below 110; and this is exactly conformable to our Rule, for multiplying $pp = 144$ by 0.756, the Product will be 108.864.

For a Proof that his Rule falls ſhort of the Truth, let us ſuppoſe $p = 45$, then $ſ$ will be $= 23$, and $3ſſ - 3ſ + 1$ will be $= 1519$, let us therefore find the Probability of the Play's terminating in that number of Games; but we have found by this LXVIII^th^ Problem, that the Probability of the Play's not terminating in that number of Games is 0.50559; and therefore the Probability of its terminating within them is 0.49441; which being leſs than $\frac{1}{2}$, ſhews 'tis not more than an equal Wager that the Play would be terminated in 1519 Games.

But farther, let us ſee what number of Games would be neceſſary for the equal wager, then multiplying 2025 ſquare of 45 by 0.756, the Product will be 1530.9; which ſhews that about 1531 Games are requiſite for it.

PROBLEM LXIX.

The ſame things being given as in the preceding Problem, except that now the ratio of a *to* b *is ſuppoſed of inequality, to ſolve the ſame by the Sines of Arcs.*

SOLUTION.

Let n repreſent the number of Games given, p the number of Stakes to be won or loſt on either ſide, let alſo A be the Semi-circumference of a Circle whoſe Radius is equal to Unity: let C, D, E, F, &c. be the Sines of the Arcs $\frac{A}{p}$, $\frac{3A}{p}$, $\frac{5A}{p}$, $\frac{7A}{p}$, &c. till the Semi-circumference be exhauſted; let alſo c, d, e, f, &c. be the reſpective verſed Sines of thoſe Arcs; let $\frac{a^n + b^n}{\overline{a+b}|^n}$ be made $= L$, $\frac{\overline{a-b}|^2}{\overline{a+b}|^2} = t$, $\frac{ab}{\overline{a+b}|^2} = r$; let $c, 2r :: CC, m$; $d, 2r :: DD, q$;

e, $2r$: : EE, ſ, &c. then the Probability of the Play not ending in n Games will be expreſſed by the following Series

$$\frac{C}{2rc+t} \times m^{\frac{1}{2}n} - \frac{D}{2rd+t} \times q^{\frac{1}{2}n} + \frac{E}{2re+t} \times ſ^{\frac{1}{2}n}, \text{ \&c.}$$

the whole to be multiplied by $\dfrac{2L}{p \times r^{\frac{1}{2}p-1}}$

As there are but few Tables of Sines, wherein the Logarithms of the verſed Sines are to be found, it will be eaſy to remedy that inconveniency, by adding the Logarithm of 2 to the exceſs of twice the tabular Logarithm of the Sine of half the given Arc above 10; for that Sum will give the Logarithm of the verſed Sine of the whole Arc.

It will be eaſily perceived that inſtead of referring the Arcs to the Diviſion of the Semi-circumference, we might have referred them to the Diviſion of the Quadrant, as in the Caſe of the preceding Problem.

Of the Summation of recurring Series.

The Reader may have perceived that the Solution of ſeveral Problems relating to Chance depends upon the Summation of Series; I have, as occaſion has offered, given the Method of ſumming them up; but as there are others that may occur, I think it neceſſary to give a ſummary View of what is moſt requiſite to be known in this matter; deſiring the Reader to excuſe me, if I do not give the Demonſtrations, which would ſwell this Tract too much; eſpecially conſidering that I have already given them in my *Miſcellanea Analytica*.

I call that a *recurring* Series which is ſo conſtituted, that having taken at pleaſure any number of its Terms, each following Term ſhall be related to the ſame number of preceding Terms, according to a conſtant law of Relation, ſuch as the following Series

$$\begin{array}{cccccc} A & B & C & D & E & F \\ 1 + & 2x + & 3xx + & 10x^3 + & 34x^4 + & 97x^5, \text{ \&c.} \end{array}$$

in which the Terms being reſpectively repreſented by the Capitals A, B, C, D, &c. we ſhall have

$$\begin{aligned} D &= 3Cx - 2Bxx + 5Ax^3 \\ E &= 3Dx - 2Cxx + 5Bx^3 \\ F &= 3Ex - 2Dxx + 5Cx^3 \\ &\text{\&c.} \end{aligned}$$

Now the Quantities $3x - 2xx + 5x^3$, taken together and connected with their proper Signs, is what I call the Index, or the *Scale of Relation*; and ſometimes the bare Coefficients $3 - 2 + 5$ are called the Scale of Relation.

PROPOSITION I.

If there be a recurring Series $a + bx + cxx + dx^3 + ex^4$, &c. of which the Scale of Relation be $fx - gxx$; the Sum of that Series continued *in infinitum* will be

$$\frac{\begin{array}{ll} a & +bx \\ & -fax \end{array}}{1 - fx + gxx}$$

PROPOSITION II.

Suppoſing that in the Series $a + bx + cxx + dx^3 + ex^4$, &c. the Law of Relation be $fx - gxx + hx^3$; the Sum of that Series continued *in infinitum* will be

$$\frac{\begin{array}{lll} a & +bx & +cxx \\ & -fax & -fbxx \\ & & +gaxx \end{array}}{1 - fx + gxx - hx^3}$$

PROPOSITION III.

Suppoſing that in the Series, $a + bx + cxx$, &c. the Law of Relation be $fx - gxx + hx^3 - kx^4$, the Sum of the Series will be

$$\frac{\begin{array}{llll} a & +bx & +cxx & +dx^3 \\ & -fax & -fbxx & -fcx^3 \\ & & +gaxx & +gbx^3 \\ & & & -hax^3 \end{array}}{1 - fx + gxx - hx^3 + kx^4}$$

As the Regularity of thoſe Sums is conſpicuous, it would be needleſs to carry them any farther.

Still it is convenient to know that the Relation being given, it will be eaſy to obtain the Sum by obſerving this general Rule.

1°, Take as many Terms of the Series as there are parts in the Scale of Relation.

2°, Subtract the Scale of Relation from Unity, and let the remainder be called the *Differential* Scale.

3°, Multiply thoſe Terms which have been taken in the Series by the Differential Scale, beginning at Unity, and ſo proceeding orderly, remembering to leave out what would naturally be extended beyond the laſt of the Terms taken.

Then the Product will be the Numerator of a Fraction expreſſing the Sum, of which the Denominator will be the Differential Scale.

Thus to form the preceding Theorem,

$$\begin{array}{rl} \text{Multiply} & a + bx + cxx + dx^3 \\ \text{by} & 1 - fx + gxx - hx^3 \ldots \end{array}$$

and beginning from Unity, we ſhall have

$$\begin{array}{llll} a & + bx & + cxx & + dx^3 \\ & - fax & - fbxx & - fcx^3 \ldots \\ & & + gaxx & + gbx^3 \ldots \\ & & & - hax^3 \ldots \end{array}$$

omitting the ſuperfluous Terms, and thus will the Numerator be formed; but the Denominator will be the Differential Scale, *viz.* $1 - fx + gxx - hx^3 + kx^4$.

COROLLARY.

If the firſt Terms of the Series are not taken at pleaſure, but begin from the ſecond Term to follow the Law of Relation, in ſo much that

$$\begin{array}{ll} b \text{ ſhall be} & = fa \\ c & = fb - ga \\ d & = fc - gb + ha \\ \text{\&c.} & \end{array}$$

then the Fraction expreſſing the Sum of the Series will have barely the firſt Term of the Series for its Numerator.

PROPOSITION IV.

If a Series is ſo conſtituted, as that the laſt Differences of the Coefficients of the Terms whereof it is compoſed be all equal to nothing, the Law of the Relation will be found in the Binomial $\overline{1-x}|^{n}$, n denoting the rank of thoſe laſt Differences; thus ſuppoſing the Series

$$\begin{array}{ccccccc} A & B & C & D & E & F & G \\ 1 & + 4x & + 10xx & + 20x^3 & + 35x^4 & + 56x^5 & + 84x^5, \text{ \&c.} \end{array}$$

whereof the Coefficients are,

$1 + 4$

	1	+ 4	+ 10	+ 20	+ 35	+ 56	+ 84
1ſt Differences		3	+ 6	+ 10	+ 15	+ 21	+ 28
2d Differences			3	+ 4	+ 5	+ 6	+ 7
3d Differences				1	+ 1	+ 1	+ 1
4th Differences					0	+ 0	+ 0

I ſay that the Relation of the Terms will be found in the Binomial $\overline{1-x}|^4$, which being expanded will be $1-4x+6xx-4x^3+x^4$ and is the Differential Scale, and therefore the Scale properly ſo called will be $4x-6xx+4x^3-x^4$; thus, in the foregoing Series, the Term

$$G = 4Fx - 6Exx + 4Dx^3 - 1Cx^4.$$

COROLLARY.

The Sums of thoſe infinite Series which begin at Unity, and have their Coefficients the figurate numbers of any order, are always expreſſible by the Fraction $\frac{1}{\overline{1-x}|^p}$, wherein p denotes the rank or order which thoſe figurative numbers obtain; for Inſtance if we take the Series $1+1x+1xx+1x^3+1x^4+1x^5+1x^6$, &c. which is a geometric Progreſſion, and whoſe Coefficients are the numbers of the firſt order, the Sum will be $\frac{1}{1-x}$, and if we take the Series $1+2x+3xx+4x^3+5x^4+6x^5+7x^6$, &c. whoſe Coefficients compoſe the numbers of the ſecond order, the Sum will be $\frac{1}{\overline{1-x}|^2}$; and again, if we take the Series $1+3x+6xx+10x^3+15x^4$, &c. whoſe Coefficients are the numbers of the third order, otherwiſe called Triangular numbers, the Sum will be $\frac{1}{\overline{1-x}|^3}$.

PROPOSITION V.

The Sum of any finite number of Terms of a recurring Series $a+bx+cxx+dx^3+ex^4$, &c. is always to be obtained.

Thus ſuppoſing the Scale of Relation to be $fx-gxx$; n the number of Terms whoſe Sum is required; and $\alpha x^n + \beta x^{n+1}$ the two Terms which would next follow the laſt of the given Terms, if the Series was continued; then the Sum will be

$$\frac{\begin{array}{l} a + bx - x^n \times \overline{\alpha + \beta x} \\ \quad - fax \qquad\qquad - f\alpha x \end{array}}{1 - fx + gxx}$$

But if the Scale of Relation be $fx - gxx + hx^3$, n the number of Terms given, and $\alpha x^n + \beta x^{n+1} + \gamma x^{n+2}$, the three Terms that would next follow the laſt of the given Terms, then the Sum will be

$$\frac{\begin{array}{lllll} a+bx & +cxx & -x^n\times & \overline{\alpha+\beta x} & \overline{+\gamma xx} \\ -fax & -fbxx & & -f\alpha x & -f\beta x \\ & +gaxx & & & +g\alpha xx \end{array}}{1-fx+gxx-hx^3}$$

The continuation of which being obvious, thoſe Theorems need not be carried any farther.

But as there is a particular elegancy for the Sums of a finite number of Terms in thoſe Series whoſe Coefficients are figurate numbers beginning at Unity, I ſhall ſet down the *Canon* for thoſe Sums.

Let n denote the number of Terms whoſe Sum is to be found, and p the rank or order which thoſe figurate numbers obtain, then the Sum will be

$$\frac{1-x^n}{\overline{1-x}^{\,p}} - \frac{nx^n}{\overline{1-x}^{\,p-1}} - \frac{n\,.\,n+1\,.\,x^n}{1\,.\,2\,.\,\overline{1-x}^{\,p-2}}$$

$$- \frac{n\,.\,n+1\,.\,n+2\,.\,x^n}{1\,.\,2\,.\,3\,.\,\overline{1-x}^{\,p-3}} - \frac{n\,.\,n+1\,.\,n+2\,.\,n+3}{1\,.\,2\,.\,3\quad 4\,.\,\overline{1-x}^{\,p-4}}, \text{ \&c.}$$

which is to be continued till the number of Terms be $= p$.

Thus ſuppoſing that the Sum of twelve Terms of the Series, $1 + 3x + 6xx + 10x^3 + 15x^4$, &c. were demanded, that Sum will be

$$\frac{1-x^{12}}{\overline{1-x}^{\,3}} - \frac{12x^{12}}{1\,.\,2\,.\,\overline{1-x}^{\,4}} - \frac{12\,.\,13x^{12}}{1\,.\,2\,.\,3\,.\,\overline{1-x}}.$$

PROPOSITION VI.

In a recurring Series, any Term may be obtained whoſe place is aſſigned.

It is very plain, from what we have ſaid, that after having taken ſo many Terms of the Series as there is in the Scale of Relation, the Series may be protracted till it reach the place aſſigned; however if that place be very diſtant from the beginning of the Series, the continuation of thoſe Terms may prove laborious, eſpecially if there be many parts in the Scale.

But there being frequent Caſes wherein that inconveniency may be avoided, it will be proper to ſhew by what Rule this may be known; and then to ſhew how we are to proceed.

The Rule will be to take the Differential Scale, and to ſuppoſe it $= 0$, then if the roots of that ſuppoſed Equation be all real, and unequal, the thing may be effected as follows. Let the Series be repreſented by

$$a + br + crr + dr^3 + er^4, \text{ \&c.}$$

and 1° if $fr - grr$ be the Scale of Relation, and conſequently $1 - fr + grr$ the differential Scale, then having made $1 - fr + grr = 0$; multiply the Terms of that Scale reſpectively by xx, x, 1, ſo as to have $xx - frx + grr = 0$, let m and p be the two roots of that Equation, then having made $A = \frac{br - pa}{m - p}$ and $B = \frac{br - ma}{p - m}$, and ſuppoſing l to be the interval between the firſt Term and the place aſſigned, that Term will be $Am^l + Bp^l$.

Secondly, If the Scale of Relation be $fr - grr + hr^3$, make $1 - fr + grr - hr^3 = 0$, the Terms of which Equation being multiplied reſpectively by x^3, xx, x, 1, we ſhall have the new Equation $x^3 - frxx + grrx - hr^3 = 0$, let m, p, q be the roots of that Equation, then having made $A = \frac{crr - \overline{p+q} \times br + pqa}{\overline{m-p} \times \overline{m-q}}$, $B = \frac{crr - \overline{m+q} \times br + mqa}{\overline{p-m} \times \overline{p-q}}$, $C = \frac{crr - \overline{p+m} \times br + mpa}{\overline{q-m} \times \overline{q-p}}$; And ſuppoſing as before l to be the Interval between the firſt Term and the Term whoſe place is aſſigned, that Term will be $Am^l + Bp^l + Cq^l$.

Thirdly, If the Scale of Relation be $fr - grr + hr^3 - kr^4$ make $1 - fr + grr - hr^3 + kr^4 = 0$, and multiply its Terms reſpectively by x^4, x^3, xx, x, 1, ſo as to have the new Equation $x^4 - frx^3 + grrx^2 - hr^3x + kr^4 = 0$, let m, p, q, $ſ$, be roots of that Equation, then having made

$$A = \frac{dr^3 - \overline{p+q+ſ} \times crr + \overline{pq+pſ+qſ} \times br - pqſ \times a}{\overline{m-p} \times \overline{m-q} \times \overline{m-ſ}}$$

$$B = \frac{dr^3 - \overline{q+ſ+m} \times crr + \overline{qſ+qm+ſm} \times br - qſm \times a}{\overline{p-q} \times \overline{p-ſ} \times \overline{p-m}}$$

$$C = \frac{dr^3 - \overline{ſ+m+p} \times crr + \overline{ſm+ſp+mp} \times br - ſmp \times a}{\overline{q-ſ} \times \overline{q-m} \times \overline{q-p}}$$

$$D = \frac{dr^3 - \overline{m+p+q} \times crr + \overline{mp+mq+pq} \times br - mpq \times a}{\overline{ſ-m} \times \overline{ſ-p} \times \overline{ſ-q}}$$

then, ſtill ſuppoſing l to be the Interval between the firſt Term and the Term whoſe place is aſſigned, that Term will be $Am^l + Bp^l + Cq^l + Dſ^l$.

Altho' one may by a narrow inſpection perceive the Order of thoſe Theorems, it will not be amiſs to expreſs them in words at length.

GENERAL RULE.

Let the Roots m, p, q, $ſ$, &c. determined as above, be called reſpectively,

ſpectively, firſt, ſecond, third, fourth Root, &c. let there be taken as many Terms of the Series beginning from the firſt, as there are parts in the Scale of Relation: then multiply in an inverted order, 1°, the laſt of theſe Terms by Unity; 2°, the laſt but one by the Sum of the Roots wanting the firſt; 3°, the laſt but two, by the Sum of the Products of the Roots taken two and two, excluding that product wherein the firſt Root is concerned; 4°, the laſt but three, by the Sum of the Products of the Roots taken three and three, ſtill excluding that Product in which the firſt Root is concerned, and ſo on; then all the ſeveral parts which are thus generated by Multiplication being connected together by Signs alternately poſitive and negative, will compoſe the Numerator of that Fraction to which A is equal; now the Numerator of that Fraction to which B is equal will be formed in the ſame manner, excluding the ſecond Root inſtead of the firſt, and ſo on

As for the Denominators, they are formed in this manner: From the firſt Root ſubtract ſeverally all the others, and let all the remainders be multiplied together, and the Product will conſtitute the Denominator of the Fraction to which A is equal; and in the ſame manner, from the ſecond Root ſubtracting all the others, let all the remainders be multiplied together, and the Product will conſtitute the Denominator of the Fraction to which B is equal, and ſo on for the Reſt.

COROLLARY I.

If the Series in which a Term is required to be aſſigned, be the Quotient of Unity divided by the differential Scale $1 - fr + grr - hr^3 + kr^4$, multiply the Terms of that Scale reſpectively by x^4, x^3, x^2, x, 1, ſo as to make the firſt Index of x equal to the laſt of r, then make the Product $x^4 - frx^3 + grrxx - hr^3x + kr^4$ to be $= 0$. Let as before m, p, q, s, be the Roots of that Equation, let alſo z be the number of thoſe Roots, and l the Interval between the firſt Term, and the Term required, then make

$$\mathrm{A} = \frac{m^{z-1}}{\overline{m-p} \times \overline{m-q} \times \overline{m-s}}, \quad \mathrm{B} = \frac{p^{z-1}}{\overline{p-m} \times \overline{p-q} \times \overline{p-s}}$$

$$\mathrm{C} = \frac{q^{z-1}}{\overline{q-m} \times \overline{q-p} \times \overline{q-s}}, \quad \mathrm{D} = \frac{s^{z-1}}{\overline{s-m} \times \overline{s-p} \times \overline{s-q}}$$

and the Term required will be $\mathrm{A}m^l + \mathrm{B}p^l + \mathrm{C}q^l + \mathrm{D}q^l$; and the Sum of the Terms will be

A ×

$$\mathrm{A}\times\frac{\overline{1-m}^{\,l+1}}{1-m}+\mathrm{B}\times\frac{\overline{1-p}^{\,l+1}}{1-p}+\mathrm{C}\times\frac{\overline{1-q}^{\,l+1}}{1-q}+\mathrm{D}\times\frac{\overline{1-ſ}^{\,l+1}}{1-ſ}$$

It is to be obſerved, that the Interval between the firſt Term and the Term required is always meaſured by the number of Terms wanting one, ſo that having for Inſtance the Terms, a, b, c, d, e, f, whereof a is the firſt and f the Term required, the Interval between a and f is 5, and the Number of all the Terms 6.

COROLLARY 2.

If in the recurring Series $a + br + crr + dr^3 + er^4$, &c. whereof the Differential Scale is ſuppoſed to be $1 - fr + grr - hr^3 + kr^4$, we make $x^4 - fxr^3 + grrxx - hr^3x + kr^4 = 0$, and that the Roots of that Equation be m, p, q, $ſ$, and that it ſo happen that ſo many Terms of the Series $a + br + crr + dr^3 + er^4$, &c. as there are Roots, be every one of them equal to Unity, then any Term of the Series may be obtained thus; let l be the Interval between the firſt Term and the Term required, make

$$\mathrm{A}=\frac{\overline{1-p}\times\overline{1-q}\times\overline{1-ſ}}{\overline{m-p}\times\overline{m-q}\times\overline{m-ſ}},\quad \mathrm{B}=\frac{\overline{1-q}\times\overline{1-ſ}\times\overline{1-m}}{\overline{p-q}\times\overline{p-ſ}\times\overline{p-m}}$$

$$\mathrm{C}=\frac{\overline{1-ſ}\times\overline{1-m}\times\overline{1-p}}{\overline{q-ſ}\times\overline{q-m}\times\overline{q-p}},\quad \mathrm{D}=\frac{\overline{1-m}\times\overline{1-p}\times\overline{1-q}}{\overline{ſ-m}\times\overline{ſ-p}\times\overline{ſ-q}}$$

and the Term required will be $\mathrm{A}m^l + \mathrm{B}p^l + \mathrm{C}q^l + \mathrm{D}ſ^l$.

PROPOSITION VII.

If there be given a recurring Series whoſe Scale of Relation is $fr - grr$, and out of that Series be compoſed two other Series, whereof the firſt ſhall contain all the Terms of the Series given which are poſited in an odd place, and the ſecond ſhall contain all the Terms that are poſited in even place; then the Scale of Relation in each of theſe two new Series may be obtained as follows:

Take the differential Scale $1 - fr + grr$, out of which compoſe the Equation $xx - frx + grr = 0$; then making $xx = z$, expunge the Quantity x, whereby the Equation will become $z - fr\sqrt{z} + grr = 0$, or $z + grr = fr\sqrt{z}$; and ſquaring both parts, to take away the Radicality, we ſhall have the new Equation $zz + 2grrz + ggr^4 = ffrrz$, or $zz + \begin{matrix}2grrz\\ -ffrrz\end{matrix} + ggr^4 = 0$; and dividing its Terms reſpectively by zz, z, 1, we ſhall have a new differential Scale for each of the two new Series into which the Series given was divided, which will be $1 + \begin{matrix}2grr\\ -ffrr\end{matrix} + ggr^4$: and this being obtained, it is plain from our *firſt Propoſition*, that each of the two new Series may be ſummed up.

But if the Scale of Relation be extended to three Terms, ſuch as the Scale $fr - grr + hr^3$, then the differential Scale for each of the two Series into which the Series given may be ſuppoſed to be divided, will be $1 \begin{array}{l} -ffrr \\ +2grr \end{array} \begin{array}{l} -2fhr^4 \\ +ggr^4 \end{array} - hhr^6$, whereby it appears that each of the two new Series may be ſummed up.

If inſtead of dividing the Series given into two Series, we divide it into three, whereof the firſt ſhall be compoſed of the 1ſt, 4th, 7th, 10th, &c. Terms; the ſecond of the 2d, 5th, 8th, 11th, &c. Terms; the third of the 3d, 6th, 9th, 12th, &c. Terms; and that the Scale of Relation be ſuppoſed $fr - grr$; then taking the differential Scale $1 - fr + grr$, and having out of it formed the Equation $xx - frx + grr = 0$, ſuppoſe $x^3 = z$; let now x be expunged, and the Equation will be changed into this $zz \begin{array}{l} +3fgr^3z \\ -f^3r^3z \end{array} + g^3r^6 = 0$, of which the Terms being divided reſpectively by zz, z, 1, we ſhall have a differential Scale $1 \begin{array}{l} -f^3r^3 \\ +3fgr^3 \end{array} + g^3r^6$, which will ſerve for every one of the three Series into which the Series given is divided; and therefore every one of thoſe three Series may be ſummed up, by help of the two firſt Terms of each.

If the Scale of Relation be compoſed of never ſo many parts, ſtill if the Series given be to be divided into three other Series; from the ſuppoſition of x^3 being made $= z$, will be derived a Scale of Relation for the three parts into which the Series given is to be divided.

But if the Series given was to be divided into 4, 5, 6, 7, &c. Series given, ſuppoſe accordingly $x^4 = z$, $x^5 = z$, $x^6 = z$, $x^7 = z$, &c. and x being expunged by the common Rules of Algebra, the Scale of Relation will be obtained for every one of the Series into which the Series given is to be divided.

PROPOSITION VIII.

If there be given two Series, each having a particular Scale of Relation, and that the correſponding Terms of both Series be added together, ſo as to compoſe a third Series, the differential Scale for this third Series will be obtained as follows.

Let $1 - fr + grr$ be the differential Scale of the firſt, and $1 - mr + prr$, the differential Scale of the ſecond; let thoſe two Scales be multiplied together, and the Product $1 - \overline{m+f} \times r + \overline{p+g+mf} \times rr - \overline{mg+pf} \times r^3 + pg \times r^4$, will expreſs the differential Scale of the Series reſulting from the addition of the other two.

And the ſame Rule will hold, if one Series be ſubtracted from the other.

PROPOSITION IX.

If there be given two recurring Series, and that the correſponding Terms of thoſe two Series be multiplied together, the differential Scale of the Series reſulting from the Multiplication of the other two may be found as follows.

Suppoſe $1 - fr + grr$ to be the differential Scale of the firſt, and $1 - ma + paa$ the differential Scale of the ſecond, ſo that the firſt Series ſhall proceed by the powers of r, and the ſecond by the powers of a; imagine thoſe two differential Scales to be Equations equal to nothing, and both r and a to be indeterminate quantities; make $ar = z$, and now by means of the three Equations, $1 - fr + grr = 0$, $1 - ma + paa = 0$, $ar = z$, let both a and r be expunged, and the Equation reſulting from that Operation will be

$$\begin{array}{llll} 1 - fmz & + ffpzz & - fgmpz^3 + ggppz^4 = 0 \\ & + mmgzz & \\ & - 2gpzz & \end{array}$$

$$\begin{array}{llll} \text{or} \quad 1 - fmar & + ffpa^2r^2 & - fgmpa^3r^3 + ggppa^4r^4 = 0 \\ & + mmga^2r^2 & \\ & - 2gpa^2r^2 & \end{array}$$

by ſubſtituting ar in the room of z; and the Terms of that Equation, without any regard to their being made $= 0$, which was purely a fiction, will expreſs the differential Scale required: and in the ſame manner may we proceed in all other more compound Caſes.

But it is very obſervable, that if one of the differential Scales be the Binomial $1 - a$ raiſed to any Power, it will be ſufficient to raiſe the other differential Scale to that Power, only ſubſtituting ar for r, or leaving the Powers of r as they are, if a be reſtrained to Unity; and that Power of the other differential Scale will conſtitute the differential Scale required.

Some Uſes of the foregoing Propoſitions.

We have ſeen in our LVIII^th^ *Problem*, that if two Adverſaries, whoſe proportion of Skill be as a to b, play together till ſuch time as either of them wins a certain number of Stakes, ſuch as 4 for inſtance, the Probability of the Play's not ending in any given number of Games will be determined by

$$\frac{4a^3b+6aabb+4ab^3}{\overline{a+b}|^4} \text{ for 4 Games.}$$
$$\frac{14a^4bb+20a^3b^3+14aab^4}{\overline{a+b}|^6} \text{ for 6 Games.}$$
$$\frac{48a^5b^3+68a^4b^4+48a^3b^5}{\overline{a+b}|^8} \text{ for 8 Games.}$$
$$\frac{164a^6b^4+232a^5b^5+164a^4b^6}{\overline{a+b}|^{10}} \text{ for 10 Games.}$$
$$\frac{560a^7b^5+732a^6b^6+560a^5b^7}{\overline{a+b}|^{12}} \text{ for 12 Games.}$$

&c.

Wherein it is evident that each Term in each of the three Columns written above is referred to the two preceding by a conſtant Scale of Relation, ſo that if the Terms of the firſt Column which are $\frac{4a^3b}{\overline{a+b}|^4}$, $\frac{14a^4bb}{\overline{a+b}|^6}$, $\frac{48a^5b^3}{\overline{a+b}|^8}$, $\frac{164a^6b^4}{\overline{a+b}|^{10}}$, $\frac{560a^7b^5}{\overline{a+b}|^{12}}$, &c. be reſpectively called E, F, G, H, K, &c. and that for ſhortneſs ſake we ſuppoſe $\frac{ab}{\overline{a+b}|^2}=r$, we ſhall find $G=4rF-2rrE$, $H=4rG-2rrF$, and ſo on; and therefore conſidering the Sum of every three Terms whereby each Probability is expreſſed as one ſingle Term, and denoting thoſe Sums reſpectively by S, T, U, X, &c. we ſhall find $U=4rT-2rrS$, $X=4rU-2rrT$, and ſo on; from which it follows that the Method of determining the Probability of the Play's not ending in any number of Games given, is no more than the finding of a Term in a recurring Series.

Let it therefore be required to find the Probability of 4 Stakes not being loſt in 60 Games, to anſwer this, let it be imagined that the Probabilities of not ending in

0, 2, 4, 6, 8, 10 - - - - - - 60 Games,

are expreſſed by C, D, E, F, G, H, - - - - - - K reſpectively; then calling l the number of Games given, it is evident that the Term K is diſtant from the Term C by an Interval $=\frac{1}{2}l$, in this Caſe $=30$, the odd numbers being omitted, by reaſon it is impoſſible an even number of Stakes ſhould be won or loſt exactly in an odd number of Games: moreover it being a certainty that the Set of 4 Stakes to be won or loſt can neither be concluded before the Play begins, nor when no more than two Games are played off, it follows that the two Terms C, and D, are each of them equal to Unity; for which reaſon, if out of the Scale of Relation $4r - 2rr$, or rather out of the differential Scale $1 - 4r + 2rr$, we form the Equation, $xx - 4rx + 2rr = 0$, and that the roots of that Equation be m and p, and then make $A = \frac{1-p}{m-p}$, $B = \frac{1-m}{p-m}$, the two Terms alone $Am^{\frac{1}{2}l} + Bp^{\frac{1}{2}l}$, will determine the Probability required. This being conformable to Corollary 2^d^ of our VI^th^ Propoſition, it will be proper to conſult it.

But becauſe in higher Caſes, that is when the number of Stakes to be won or loſt is larger, it would ſometimes be infinitely laborious to extract the Roots of thoſe Equations, it will be proper to ſhew how thoſe Roots are actually to be found in a Table of Sines. Of which to give one Inſtance, let it be propoſed to find the Probability of the Play's not ending in any number of Games l, when the number of Stakes to be won or loſt is 6; then arguing in the ſame manner as in the preceding Caſe, let the Probabilities of the
Play's not being concluded in 0, 2, 4, 6, 8, 10 - - - - l Games
be reſpectively D, E, F, G, H, K - - - - z; then
we may conclude that the three Terms D, E, F ſtanding reſpectively over-againſt the number of Games 0, 2, 4, are each of them equal to Unity, it being a certainty that the Play cannot be concluded in that number of Games. Wherefore having taken the differential Scale $1 - 6r + 9rr - 2r^3$, which belongs to that number of Stakes 6, and formed out of it the Equation $x^3 - 6rxx + 9rrx - 2r^3 = 0$, let the Roots of that Equation be denoted by m, p, q; then making $A = \frac{\overline{1-p} \times \overline{1-q}}{\overline{m-p} \times \overline{m-q}}$, $B = \frac{\overline{1-q} \times \overline{1-m}}{\overline{p-q} \times \overline{p-m}}$, $C = \frac{\overline{1-m} \times \overline{1-p}}{\overline{q-m} \times \overline{q-p}}$, the Probability required will be $Am^{\frac{1}{2}l} + Bp^{\frac{1}{2}l} + Cq^{\frac{1}{2}l}$.

Now I ſay that the Roots m, p, q of the Equation above written, may be derived from a Table of Sines; for if the Semi-circumference of a Circle whoſe Radius is $2r$, be divided into 6 equal parts, and we take the Co-verſed Sines of the Arcs that are $\frac{1}{6}$, $\frac{3}{6}$, $\frac{5}{6}$ of the Semi-circumference, ſo that the Numerators of thoſe Fractions be all the odd numbers contained in 6, thoſe Co-verſed Sines will be the Values of m, p, q, and the Rule is general and extends to all Caſes; ſtill it is obſervable that when the number of Stakes is odd, for Inſtance 9, we ought to take only $\frac{1}{9}, \frac{3}{9}, \frac{5}{9}, \frac{7}{9}$ of the Semi-circumference, and reject the laſt Term $\frac{9}{9}$ expreſſing the whole Semi-circumference.

But what ought chiefly to recommend this Method is, that ſuppoſing m to be the greateſt Co-verſed Sine, the firſt Term alone $Am^{\frac{1}{2}l}$ will give a ſufficient approximation to the Probability required, eſpecially if l be a large number in itſelf, and it be alſo large in reſpect to the number of Stakes.

Still theſe Rules would not be eaſily practicable by reaſon of the great number of Factors which might happen to be both in the Numerator and Denominator to which A is ſuppoſed equal, if I had not, from a thorough inſpection into the nature of the Equations which determine the Values of m, p, q, &c. deduced the following Theorems.

1°, If n repreſents the number of Stakes to be won or loſt, whether that number be even or odd, then the Numerator of the Fraction to which A is equal, *viz.* $\overline{1-p} \times \overline{1-q} \times \overline{1-s} \times \overline{1-t}$, &c. will always be equal to the Fraction $\frac{a^n + b^n}{\overline{a+b}|^n \times \overline{1-m}}$; and in the ſame manner that the Numerator of the Fraction to which B is equal, *viz.* $\overline{1-q} \times \overline{1-s} \times \overline{1-t}$, &c. will always be equal to the Fraction $\frac{a^n + b^n}{\overline{a+b}|^n \times \overline{1-p}}$, and ſo on.

2°, If n be an even number, and that m' be the right Sine correſponding to the Co-verſed Sine m; then the Denominator of the Fraction to which A is equal, *viz.* $\overline{m-p} \times \overline{m-q} \times \overline{m-s} \times \overline{m-t}$, &c. will always be equal to the Fraction $\frac{nr^{\frac{1}{2}n}}{m'}$; and in the ſame manner if p' repreſent the right Sine belonging to the Co-verſed Sine p, then the Denominator of the Fraction to which B is equal, *viz.* $\overline{p-q} \times \overline{p-s} \times \overline{p-t}$, &c. will always be equal to the Fraction $\frac{nr^{\frac{1}{2}n}}{p'}$, and ſo on.

3°, If n be an odd number, and that m' be, as before, the right Sine correſponding to the Co-verſed Sine m; then the Denominator of the Fraction to which A is equal will be $\frac{nr^{\frac{1}{2}n}}{m'\sqrt{m}}$, and the Denominator of the Fraction to which B is equal will be $\frac{nr^{\frac{1}{2}n}}{p'\sqrt{p}}$.

COROLLARY

From all which it follows, that the Method of determining the Probability of a certain number n of Stakes not being loſt in a given number l of Games, may be thus expreſſed.

Let L be ſuppoſed $= \frac{a^n + b^n}{\overline{a+b}|^n}$, and $r = \frac{ab}{\overline{a+b}|^2}$, then that Probability will be

$$\frac{L}{nr^{\frac{1}{2}n}} \text{ into } \frac{m'}{1-m}\times m^{\frac{1}{2}l} - \frac{p'}{1-p}\times p^{\frac{1}{2}l} + \frac{q'}{1-p}\times q^{\frac{1}{2}l} - \frac{s'}{1-s}\times s^{\frac{1}{2}l}$$

&c. when n is an even number, or

$$\frac{L}{nr^{\frac{1}{2}n}} \text{ into } \frac{m'\sqrt{m}}{1-m}\times m^{\frac{l-1}{2}} - \frac{p'\sqrt{p}}{1-p}\times p^{\frac{l-1}{2}} + \frac{q'\sqrt{q}}{1-q}\times q^{\frac{l-1}{2}} - \frac{s\sqrt{s}}{1-s}\times s^{\frac{l-1}{2}}$$

&c. when n is an odd nnmber.

But becauſe $m^{\frac{l-1}{2}}\times\sqrt{m}$, $p^{\frac{l-1}{2}}\times\sqrt{p}$, &c. are the ſame as $m^{\frac{1}{2}l}$, $p^{\frac{1}{2}l}$ reſpectively, it is plain that both Caſes are reduced to one and the ſame Rule.

It was upon this foundation that I preſcribed the Rule to be ſeen in my LXIXth Problem, wherein I did not diſtinguiſh the odd Caſes from the even.

But altho' the Rule there given ſeems ſomewhat different from what it is here, yet at bottom there is no difference; it conſiſting barely in this, that whereas $2r$ in this place is the Radius of the Circle to which the Calculation is adapted, there it is *Unity*, and that there the Co-verſed Sines were expreſſed by their Equivalents in right Sines; there was alſo this little difference, that the Denominators $1-m$, $1-p$, &c. were expreſſed by means of the verſed Sines of thoſe Arcs, to which m and p are co-verſed Sines.

Other Variations might be introduced, ſuch for inſtance as might ariſe from the conſideration of $\sqrt{mr}$, $\sqrt{pr}$, &c. being the right

Sines of $\frac{1}{2}$ the Complements to a Quadrant of the Arcs originally taken.

But to ſhew the farther uſe of theſe Series, it will be convenient to propoſe a Problem or two more relating to that Subject.

PROBLEM LXX.

M *and* N, *whoſe proportion of Chances to win one Game are reſpectively as* a *to* b, *reſolve to play together till one or the other has loſt* 4 *Stakes: two Standers by,* R *and* S, *concern themſelves in the Play,* R *takes the ſide of* M, *and* S *of* N, *and agree betwixt them, that* R *ſhall ſet to* S, *the Sum* L *to the Sum* G *on the firſt Game,* 2L *to* 2G *on the ſecond,* 3L *to* 3G *on the third,* 4L *to* 4G *on the fourth, and in caſe the Play be not then concluded,* 5L *to* 5G *on the fifth, and ſo increaſing perpetually in Arithmetic Progreſſion the Sums which they are to ſet to one another, as long as* M *and* N *play; yet with this farther condition, that the Sums, ſet down by them* R *and* S, *ſhall at the end of each Game be taken up by the Winner, and not left upon the Table to be taken up at once upon the Concluſion of the Play: it is demanded how the Gain of* R *is to be eſtimated before the Play begins.*

SOLUTION.

Let there be ſuppoſed a time wherein the number p of Games has been played; then R having the number a of Chances to win the Sum $\overline{p+1}\times G$ in the next Game; and S having the number b of Chances to win the Sum $\overline{p+1}\times L$, it is plain that the Gain of R in that circumſtance ought to be eſtimated by the quantity $\overline{p+1}\times\frac{aG-bL}{a+bL}$; but this Gain being to be eſtimated before the Play begins, it follows that it ought to be eſtimated by the quantity $\overline{p+1}\times\frac{aG-bL}{a+b}$ multiplied by the reſpective Probability there is that the Play will not then be ended; and therefore the whole Gain of *R* is the Sum of the Probabilities of the Play's not ending in 0, 1, 2, 3, 4, 5, 6, &c. *Games in infinitum*, multiplied by the reſpective Values of the quantity $\overline{p+1}\times\frac{aG-bL}{a+b}$, p being interpreted ſucceſſively by the Terms of the Arithmetic Progreſſion, 0, 1, 2, 3, 4, 5, 6, &c. Now, let theſe Probabilities of the Play's not ending be reſpectively repreſented by A, B, C, D, E, F, G, I, &c. let alſo the Quantity $\frac{aG-bL}{a+b}$ be called S, and then it will follow that the Gain of *R* will be expreſſed by the Series $AS+2BS+3CS+4DS+5ES+6FS+7GS$, &c. but in this Problem, altho' the Probabilities of the Play's not ending decreaſe continually, yet the number of Stakes being even, the Probability of the Play's not ending in an odd number of Games is not leſs than the Probability of not ending in the even number that immediately precedes the odd; and therefore $B=A$, $D=C$, $F=E$, $I=G$, &c. from whence it follows that the Gain of *R* will be expreſſed by the product of S *into* $3A+7C+11E+15G+19I$, &c. but the differential Scale for the Series $A+C+E+G$, &c. is $1-4r+2rr$, wherein r is ſuppoſed $=\frac{ab}{\overline{a+b}\rceil^2}$, and the differential Scale for the Series $3+7+11+15+19$, &c. is $1-3a+3aa-a^3$, wherein $a=1$. And therefore the differential Scale for the Series $3A+7C+11E$, &c. conſiſting of the products of the Terms of one Series by the correſponding Terms of the other, will be $\overline{1-4r+2rr}\rceil^2$, or $1-8r+20rr-16r^3+4r^4$; and therefore having written down the four firſt Terms of the Series to be ſummed up, *viz.* as many Terms wanting one as there are in the differential Scale, multiply them in order by the differential Scale according to the preſcription given in the Remark belonging to our third Propoſition, and the Product will be the Numerator of the Fraction expreſſing the Sum, of which Fraction the Denominator will be $\overline{1-4r+2rr}\rceil^2$; But to make this the plainer, here follows the Operation,

$$\begin{array}{llll}
3A+ & 7C+ & 11E+ & 15G \\
1- & 8r+ & 20rr- & 16r^3\ldots\ldots \\
\hline
3A+ & 7C+ & 11E+ & 15G \\
 & -24rA- & 56rC- & 88rE\,. \\
 & & +60rrA+ & 140rrC\,.\,. \\
 & & & -48r^3A
\end{array}$$

And thus is the Numerator obtained: but $A = 1$, it being a certainty that the Play cannot be ended before it is begun, and C is likewiſe $= 1$, it being a certainty that 4 Stakes cannot be loſt neither before nor at the expiration of 2 Games; but by the law of Relation of the Terms of the Series, $E = 4rC - 2rrA$, and $G = 4rE - 2rrC$, and therefore the proper Subſtitutions being made, the Sum of the Series will be found to be S *into* $\frac{10 - 36r + 36rr + 8r^3}{\overline{1 - 4r + 2rr}|^2}$ and now in the room of S and r ſubſtituting their reſpective Values $\frac{aG - bL}{a+b}$ and $\frac{ab}{\overline{a+b}|^2}$ the Sum $\frac{aG - bL}{a+b}$ *into*

$$\frac{10 + 24a^5b + 42a^4bb + 64a^3b^3 + 42aab^4 + 24ab^5 + 10b^6 \times \overline{a+b}|^2}{\overline{a^4 + b^4}|^2}$$

will expreſs the Gain of *R*.

COROLLARY 1.

If the Stake L be greater than the Stake G, in the ſame proportion as a is greater than b, there can be no advantage on either ſide.

COROLLARY 2.

If a and b are equal, the Gain of *R* will be 216 times the half difference between the Stakes G and L: thus if G ſtands for a Guinea of $21^{sh.}$ and L for $20^{sh.}$ the Gain of *R* will be 216 Sixpences, that is, $5^{L.} - 8^{sh.}$

COROLLARY 3.

If a be greater than b, the Gain of *R*, according to that inequality, will vary an infinite number of ways, yet not be greateſt when the proportion of a to b is greateſt; ſo that for Inſtance, if the proportion of a to b is 2 to 1, and G and L are equal, the Gain of *R* will be about $29\frac{1}{4}G$; but if a is to b as 3 to 1, the Gain of *R* will be no more than about $22\frac{1}{4}G$; and if the proportion of a to b be infinitely great, which would make *R* win infallibly, the Gain of *R* will be only 10 G. But altho' this may ſeem at firſt a very ſtrange Paradox, yet the reaſon of it will eaſily be apprehended from this conſideration, that the greater the proportion is of a to b, ſo much the ſooner is the Play likely to be concluded; and therefore if that proportion were infinite, the Play would neceſſarily be terminated in 4 Games, which would make the Gain of *R* to be $1 + 2 + 3 + 4 = 10$.

But if it was required what muſt be the proportion of *a* to *b* which will afford to *R* the greateſt advantage poſſible, the anſwer will be very near 2 to 1, as may be found eaſily upon Trial; and may be found accurately by the Method which the Geometricians call *de Maximis & Minimis.*

PROBLEM LXXI.

If M *and* N, *whoſe number of Chances to win one Game are reſpectively as* a *to* b, *play together till four Stakes are won or loſt on either ſide; and that at the ſame time,* R *and* S *whoſe number of Chances to win one Game are reſpectively as* c *to* d, *play alſo together till five Stakes are won or loſt on either ſide; what is the Probability that the Play between* M *and* N *will be ended in fewer Games, than the Play between* R *and* S.

SOLUTION.

The Probability of the firſt Play's being ended in any number of Games before the ſecond, is compounded of the Probability of the firſt Play's being ended in that number of Games, and of the ſecond's not being ended with the Game immediately preceding: from whence it follows, that the Probability of the firſt Play's ending in an indeterminate number of Games before the ſecond, is the Sum of all the Probabilities *in infinitum* of the firſt Play's ending, multiplied by the reſpective Probabilities of the ſecond's not being ended with the Game immediately preceding.

Let A, B, C, D, E, &c. repreſent the Probabilities of the firſt Play's ending in 4, 6, 8, 10, 12, &c. Games reſpectively; let alſo F, G, H, K, L, &c. repreſent the Probabilities of the ſecond's not being ended in 3, 5, 7, 9, 11, &c. Games reſpectively: hence, by what we have laid down before, the Probability of the firſt Play's ending before the ſecond will be repreſented by the infinite Series AF + BG + CH + DK + EL, &c. Now to find the Law of Relation in this third Series, we muſt fix the Law of Relation in the firſt and ſecond, which will be done by our LXth Problem, it being for the firſt $4r - 2rr$, wherein r is ſuppoſed $= \frac{ab}{\overline{a+b}|^2}$; and becauſe, as we have obſerved before, the Law of Relation in thoſe

Series which expreſs the Probability of not ending, is the ſame as the Law of Relation in the reſpective Series which expreſs the Probability of ending; it will alſo be found by the directions given in our LX^th^ Problem, that if we ſuppoſe $\frac{cd}{\overline{c+d}|^2} = m$, the Law of Relation for the ſecond Series will be $5m - 5mm$, and therefore the Laws of Relation in the firſt and ſecond Series will reſpectively be $1 - 4r + 2rr$, $1 - 5m + 5mm$. And now having ſuppoſed thoſe two differential Scales as Equations $= 0$, and ſuppoſed alſo $rm = z$, we ſhall find by the Rules delivered in our IX^th^ Propoſition, that the Scale of Relation for the third Series will be $1 - 20z + 110zz - 200z^3 + 100z^4$; and therefore having taken the four firſt Terms of the third Series, and multiplied them by the differential Scale, according to the proper Limitations preſcribed in our III^d^ Propoſition, we ſhall find the Sum of the third Series to be

$$\frac{\begin{array}{llll} AF + BG & + CH & + DK \\ & - 20AFGz - 20BGz & - 20CHz \\ & & + 110AFzz + 110BGzz \\ & & - 200AFz^3 \end{array}}{1 - 20z + 110z^2 - 200z^3 + 100z^4}$$

Now ſuppoſing S to repreſent the Fraction $\frac{a^4 + b^4}{\overline{a+b}|^4}$, the four Terms A, B, C, D will be found to be $1S + 4rS + 14rrS + 48r^3S$; but the four Terms F, G, H, K wherein S is not concerned will be found to be 1, $5m - 5mm$, $20mm - 25m^3$, $75m^3 - 100m^4$; and therefore the proper Subſtitutions being made in the Sum above written, we ſhall have that Sum reduced to its proper *Data*; and that Sum thus reduced will exhibit the Probability required. But becauſe thoſe *Data* are many, it cannot be expected that the Solution ſhould have ſo great a degree of Simplicity as if we had reſtrained *a* and *b* to a ratio of Equality, which if we had, the Probability required would have been expreſſed by the Fraction $\frac{2z - 10zz + cz^3}{1 - 20z + 110zz - 200z^3 + 100z^4}$; but becauſe r has been ſuppoſed $= \frac{ab}{\overline{a+b}|^2}$, it follows that r in this Caſe is $= \frac{1}{4}$: and again, becauſe m has been ſuppoſed $= \frac{cd}{\overline{c+a}|^2}$, then m is alſo $= \frac{1}{4}$, for which reaſon rm or $z = \frac{1}{16}$, for which reaſon ſubſtituting $\frac{1}{16}$ inſtead of z, the Probability required will be expreſſed by the

Fraction $\frac{476}{723}$: Now ſubtracting this Fraction from Unity, the remainder will be the Fraction $\frac{247}{723}$, and therefore the Odds of the firſt Play's ending before the ſecond will be 476 to 247, or 27 to 14 nearly.

PROBLEM LXXII.

A *and* B *playing together, and having an equal number of Chances to win one Game, engage to a Spectator* S *that after an even number of Games* n *is over, the Winner ſhall give him as many Pieces as he wins Games over and above one half the number of Games played, it is demanded how the Expectation of* S *is to be determined.*

SOLUTION.

Let E denote the middle Term of the Binomial $a+b$ raiſed to the Power n, then $\frac{\frac{1}{2}nE}{2^n}$ will expreſs the number of Pieces which the Spectator has a right to expect.

Thus ſuppoſing that *A* and *B* were to play 6 Games, then raiſing $a+b$ to the 6th Power, all the following Terms will be found in it, *viz.* $a^6+6a^5b+15a^4bb+20a^3b^3+15aab^4+6ab^5+b^6$.

But becauſe the Chances which *A* and *B* have to win one Game have been ſuppoſed equal, then a and b may both be made $=1$, which will make it that the middle Term E will be 20; therefore this number being multiplied by $\frac{1}{2}n$, that is in this Caſe by 3, the Product will be 60, which being divided by 2^n or 2^6, that is by 64, the Quotient will be $\frac{60}{64}$ or $\frac{15}{16}$, and therefore the Expectation of *S* is as good to him as if he had $\frac{15}{16}$ of a Piece given him, and for that Sum he might transfer his Right to another.

It will be eaſy by Trial to be ſatisfied of the Truth of this Concluſion, for reſuming the 6th Power of $a+b$, and conſidering the firſt Term a^6, which ſhews the number of Chances for *A* to win 6 times; in which Caſe *S* would have 3 Pieces given him, then the

Expectation of *S* ariſing from that proſpect is $\frac{3a^6}{\overline{a+b}|^6}$, that is $\frac{3}{64}$; conſidering next the Term $6a^5b$ which denotes the number of Chances for *A* to win 5 times and loſing once, whereby he would get two Games above 3, and conſequently *S* get 2 Pieces, then the Expectation of *S* ariſing from that proſpect would be $\frac{2 \times 6a^5b}{\overline{a+b}|^6}$ or $\frac{12}{64}$; laſtly conſidering the third Term $15a^4b^2$ which ſhews the number of Chances for *A* to get 4 Games out of 6, and conſequently for *S* to get 1 Piece, the Expectation of *S* ariſing from that proſpect would be $\frac{1 \times 15a^4bb}{\overline{a+b}|^4}$ or $\frac{15}{64}$, the fourth Term $20a^3b^3$ would afford nothing to *S*, it denoting the number of Chances for *A* to win no more than 3 Games; and therefore that part of the Expectation of *S*, which is founded on the Engagement of *A* to him, would be $\frac{3+12+15}{64} = \frac{30}{64}$; but he expects as much from *B*, and therefore his whole Expectation is $\frac{60}{64} = \frac{15}{16}$ as had been before determined.

And in the ſame manner, if *A* and *B* were to play 12 Games the Expectation of *S* would be $\frac{5544}{4096}$, which indeed is greater than in the preceding Caſe, but leſs than in the proportion of the number of Games played, his Expectation in this Caſe being to the former as 5544 to 3840, which is very little more than in the proportion of 3 to 2, but very far from the proportion of 12 to 6, or 2 to 1.

And if we ſuppoſe ſtill a greater number of Games to be played between *A* and *B*, the Expectation of *S* would ſtill increaſe, but in a leſs proportion than before; for inſtance, if *A* and *B* were to play 100 Games, the Expectation of *S* would be 3.9795; if 200, 5.6338; if 300, 6.9041; if 400, 7.9738; if 500, 8.9161; if 700, 800, 900, 10.552, 11.280, 11.965 reſpectively, ſo that in 100 Games the Expectation of *S* would be in reſpect to that number of Games about $\frac{1}{25}$, and in 900 Games that Expectation would not be above $\frac{1}{75}$. Now how to find the middle Terms of thoſe high Powers will be ſhewn afterwards.

COROLLARY.

From the foregoing conſiderations, it follows, that if after taking a great number of Experiments, it ſhould be obſerved that the happenings or failings of an Event have been very near a ratio of Equality, it may ſafely be concluded, that the Probabilities of its happening or failing at any one time aſſigned are very near equal.

PROBLEM LXXIII.

A *and* B *playing together, and having a different number of Chances to win one Game, which number of Chances I ſuppoſe to be reſpectively as* a *to* b, *engage themſelves to a Spectator* S, *that after a certain number of Games is over,* A *ſhall give him as many Pieces as he wins Games, over and above* $\frac{a}{a+b}n$, *and* B *as many as he wins Games, over and above the number* $\frac{b}{a+b}n$; *to find the Expectation of* S.

SOLUTION.

Let E be that Term of the Binomial $a+b$ raiſed to the Power n, in which the Indices of the Powers of a and b ſhall be in the ſame ratio to one another as a is to b; let alſo p and q denote reſpectively thoſe Indices, then will the Expectation of *S* from *A* and *B* together be $\frac{2pq}{n \times \overline{a+b}|^n}$E, or $\frac{pq}{n \times \overline{a+b}|^n}$E from either of them in particular.

Thus ſuppoſing the number of Games n to be 6, and that the ratio of a to b is as 2 to 1; then that Term E of the Binomial $a+b$ raiſed to its 6th Power, wherein the Indices have the ſame ratio to one another as 2 to 1, is $15a^4b^2$, and therefore $p=4$, and $q=2$; and becauſe, a, b, p, q, n are reſpectively 2, 1, 4, 2, 6, thence the Expectation $\frac{2pq}{n \times \overline{a+b}|^n} \times$E will be in this particular Caſe $\frac{16}{4374} \times 240$, or $\frac{640}{729} = \frac{9}{10}$ nearly.

But ſuppoſing that *A* and *B* reſolve to play 12 Games, then that Term of the Binomial $a+b$ raiſed to its 12th Power, wherein the Indices p and q have the ſame ratio as 2 to 1, is $495a^8b^4$; and becauſe the Quantities a, b, p, q, n, are reſpectively 2, 1, 8, 4, 12, the Expectation of *S* will be $\frac{675840}{531441}$ or $\frac{14}{11}$ nearly.

And again, if *A* and *B* play ſtill a greater number of Games, the Expectation of *S* will perpetually increaſe, but in a leſs proportion than of the number of Games played.

COROLLARY.

From this it follows, that if after taking a great number of Experiments, it ſhould be perceived that the happenings and failings have been nearly in a certain proportion, ſuch as of 2 to 1, it may ſafely be concluded that the Probabilities of happening or failing at any one time aſſigned will be very near in that proportion, and that the greater the number of Experiments has been, ſo much nearer the Truth will the conjectures be that are derived from them.

But ſuppoſe it ſhould be ſaid, that notwithſtanding the reaſonableneſs of building Conjectures upon Obſervations, ſtill conſidering the great Power of Chance, Events might at long run fall out in a different proportion from the real Bent which they have to happen one way or the other; and that ſuppoſing for Inſtance that an Event might as eaſily happen as not happen, whether after three thouſand Experiments it may not be poſſible it ſhould have happened two thouſand times and failed a thouſand; and that therefore the Odds againſt ſo great a variation from Equality ſhould be aſſigned, whereby the Mind would be the better diſpoſed in the Concluſions derived from the Experiments.

In anſwer to this, I'll take the liberty to ſay, that this is the hardeſt Problem that can be propoſed on the Subject of Chance, for which reaſon I have reſerved it for the laſt, but I hope to be forgiven if my Solution is not fitted to the capacity of all Readers; however I ſhall derive from it ſome Concluſions that may be of uſe to every body: in order thereto, I ſhall here tranſlate a Paper of mine which was printed *November* 12, 1733, and communicated to ſome Friends, but never yet made public, reſerving to myſelf the right of enlarging my own Thoughts, as occaſion ſhall require.

Novemb. 12, 1733.

A Method of approximating the Sum of the Terms of the Binomial $\overline{a+b}|^n$ *expanded into a Series, from whence are deduced ſome practical Rules to eſtimate the Degree of Aſſent which is to be given to Experiments.*

ALTHO' the Solution of Problems of Chance often requires that ſeveral Terms of the Binomial $\overline{a+b}|^n$ be added together, nevertheleſs in very high Powers the thing appears ſo laborious, and of ſo great difficulty, that few people have undertaken that Taſk; for beſides *James* and *Nicolas Bernoulli*, two great Mathematicians, I know of no body that has attempted it; in which, tho' they have ſhewn very great ſkill, and have the praiſe which is due to their Induſtry, yet ſome things were farther required; for what they have done is not ſo much an Approximation as the determining very wide limits, within which they demonſtrated that the Sum of the Terms was contained. Now the Method which they have followed has been briefly deſcribed in my *Miſcellanea Analytica*, which the Reader may conſult if he pleaſes, unleſs they rather chuſe, which perhaps would be the beſt, to conſult what they themſelves have writ upon that ſubject: for my part, what made me apply myſelf to that Inquiry was not out of opinion that I ſhould excel others, in which however I might have been forgiven; but what I did was in compliance to the deſire of a very worthy Gentleman, and good Mathematician, who encouraged me to it: I now add ſome new thoughts to the former; but in order to make their connexion the clearer, it is neceſſary for me to reſume ſome few things that have been delivered by me a pretty while ago.

I. It is now a dozen years or more ſince I had found what follows; If the Binomial $1+1$ be raiſed to a very high Power denoted by n, the ratio which the middle Term has to the Sum of all the Terms, that is, to 2^n, may be expreſſed by the Fraction $\frac{2A \times \overline{n-1}|^n}{n^n \times \sqrt{n-1}}$, wherein A repreſents the number of which the Hyperbolic Logarithm is $\frac{1}{12} - \frac{1}{360} + \frac{1}{1260} - \frac{1}{1680}$, &c. But becauſe the Quantity $\frac{\overline{n-1}^n}{n^n}$ or $\overline{1-\frac{1}{n}}^n$ is very nearly given when n is a high Power, which is not difficult to prove, it follows that, in an infinite Power, that Quantity will be abſolutely given, and repreſent the number of which the Hyperbolic Logarithm is -1; from whence it follows, that if B denotes the Number of which the Hyperbolic Logarithm is $-1+\frac{1}{12}-\frac{1}{360}+\frac{1}{1260}-\frac{1}{1680}$, &c. the Expreſſion above-written will become $\frac{2B}{\sqrt{n-1}}$ or barely $\frac{2B}{\sqrt{n}}$: and that therefore if we change the Signs of that Series, and now ſuppoſe that B repreſents the Number of which the Hyperbolic Logarithm is $1-\frac{1}{12}+\frac{1}{360}-\frac{1}{1260}+\frac{1}{1680}$, &c. that Expreſſion will be changed into $\frac{2}{B\sqrt{n}}$.

When I firſt began that inquiry, I contented myſelf to determine at large the Value of B, which was done by the addition of ſome Terms of the above-written Series; but as I perceived that it converged but ſlowly, and ſeeing at the ſame time that what I had done anſwered my purpoſe tolerably well, I deſiſted from proceeding farther till my worthy and learned Friend Mr. *James Stirling*, who had applied himſelf after me to that inquiry, found that the Quantity B did denote the Square-root of the Circumference of a Circle whoſe Radius is Unity, ſo that if that Circumference be called c, the Ratio of the middle Term to the Sum of all the Terms will be expreſſed by $\frac{2}{\sqrt{nc}}$.

But altho' it be not neceſſary to know what relation the number B may have to the Circumference of the Circle, provided its value be attained, either by purſuing the Logarithmic Series before mentioned, or any other way; yet I own with pleaſure that this diſcovery, beſides that it has ſaved trouble, has ſpread a ſingular Elegancy on the Solution.

II. I alſo found that the Logarithm of the Ratio which the middle Term of a high Power has to any Term diſtant from it by an Interval denoted by l, would be denoted by a very near approximation, (ſuppoſing $m=\frac{1}{2}n$) by the Quantities $\overline{m+l-\frac{1}{2}}\times$ Log. $\overline{m+l-1}+\overline{m-l+\frac{1}{2}}\times$ Log. $\overline{m-l+1}-2m\times$ Log. $m+$ Log. $\frac{m+l}{m}$

COROL-

COROLLARY 1.

This being admitted, I conclude, that if m or $\frac{1}{2}n$ be a Quantity infinitely great, then the Logarithm of the Ratio, which a Term diſtant from the middle by the Interval l, has to the middle Term, is $-\frac{2ll}{n}$.

COROLLARY 2.

The Number, which anſwers to the Hyperbolic Logarithm $-\frac{2ll}{n}$, being

$$1-\frac{2ll}{n}+\frac{4l^4}{2nn}-\frac{8l^6}{6n^3}+\frac{16l^8}{24n^4}-\frac{32l^{10}}{120n^5}+\frac{64l^{12}}{720n^6}, \text{ \&c.}$$

it follows, that the Sum of the Terms intercepted between the Middle, and that whoſe diſtance from it is denoted by l, will be

$$\frac{2}{\sqrt{nc}} \text{ into } l-\frac{2l^3}{1\times 3n}+\frac{4l^5}{2\times 5nn}-\frac{8l^7}{6\times 7n^3}+\frac{16l^9}{24\times 9n^4}-\frac{32l^{11}}{120\times 11n^5}, \text{ \&c.}$$

Let now l be ſuppoſed $=s\sqrt{n}$, then the ſaid Sum will be expreſſed by the Series

$$\frac{2}{\sqrt{c}} \text{ into } ſ-\frac{2ſ^3}{3}+\frac{4ſ^5}{2\times 5}-\frac{8ſ^7}{6\times 7}+\frac{16ſ^9}{24\times 9}-\frac{32ſ^{11}}{120\times 11}, \text{ \&c.}$$

Moreover, if ſ be interpreted by $\frac{1}{2}$, then the Series will become

$$\frac{2}{\sqrt{c}} \text{ into } \frac{1}{2}-\frac{1}{3\times 4}+\frac{1}{2\times 5\times 8}-\frac{1}{6\times 7\times 10}+\frac{1}{24\times 9\times 32}-\frac{1}{120\times 11\times 64}, \text{ \&c.}$$

which converges ſo faſt, that by help of no more than ſeven or eight Terms, the Sum required may be carried to ſix or ſeven places of Decimals: Now that Sum will be found to be 0.427812, independently from the common Multiplicator $\frac{2}{\sqrt{c}}$, and therefore to the Tabular Logarithm of 0.427812, which is 9.6312529, adding the Logarithm of $\frac{2}{\sqrt{c}}$, *viz.* 9.9019400, the Sum will be 19.5331929, to which anſwers the number 0.341344.

LEMMA.

If an Event be ſo dependent on Chance, as that the Probabilities of its happening or failing be equal, and that a certain given number n of Experiments be taken to obſerve how often it happens and fails, and alſo that l be another given number, leſs than $\frac{1}{2}n$, then the Probability of its neither happening more frequently than $\frac{1}{2}n+l$

times, nor more rarely than $\frac{1}{2}n - l$ times, may be found as follows.

Let L and L be two Terms equally diſtant on both ſides of the middle Term of the Binomial $\overline{1+1}^n$ expanded, by an Interval equal to l; let alſo $ſ$ be the Sum of the Terms included between L and L together with the Extreams, then the Probability required will be rightly expreſſed by the Fraction $\frac{ſ}{2^n}$; which being founded on the common Principles of the Doctrine of Chances, requires no Demonſtration in this place.

COROLLARY 3.

And therefore, if it was poſſible to take an infinite number of Experiments, the Probability that an Event which has an equal number of Chances to happen or fail, ſhall neither appear more frequently than $\frac{1}{2}n + \frac{1}{2}\sqrt{n}$ times, nor more rarely than $\frac{1}{2}n - \frac{1}{2}\sqrt{n}$ times, will be expreſſed by the double Sum of the number exhibited in the ſecond Corollary, that is, by 0.682688, and conſequently the Probability of the contrary, which is that of happening more frequently or more rarely than in the proportion above aſſigned will be 0.317312, thoſe two Probabilities together compleating Unity, which is the meaſure of Certainty: Now the Ratio of thoſe Probabilities is in ſmall Terms 28 to 13 very near.

COROLLARY 4.

But altho' the taking an infinite number of Experiments be not practicable, yet the preceding Concluſions may very well be applied to finite numbers, provided they be great: for Inſtance, if 3600 Experiments be taken, make $n = 3600$, hence $\frac{1}{2}n$ will be $= 1800$, and $\frac{1}{2}\sqrt{n} = 30$, then the Probability of the Event's neither appearing oftner than 1830 times, nor more rarely than 1770, will be 0.682688.

COROLLARY 5.

And therefore we may lay this down for a fundamental Maxim, that in high Powers, the Ratio, which the Sum of the Terms included between two Extreams diſtant on both ſides from the middle Term by an Interval equal to $\frac{1}{2}\sqrt{n}$, bears to the Sum of all the Terms, will be rightly expreſſed by the Decimal 0.682688, that is $\frac{28}{41}$ nearly.

Still, it is not to be imagined that there is any neceſſity that the number n ſhould be immenſely great; for ſuppoſing it not to reach beyond the 900th Power, nay not even beyond the 100th, the Rule here given will be tolerably accurate, which I have had confirmed by Trials.

But it is worth while to obſerve, that ſuch a ſmall part as is $\frac{1}{2}\sqrt{n}$ in reſpect to n, and ſo much the leſs in reſpect to n as n increaſes, does very ſoon give the Probability $\frac{28}{41}$ or the Odds of 28 to 13; from whence we may naturally be led to enquire, what are the Bounds within which the proportion of Equality is contained? I anſwer, that theſe Bounds will be ſet at ſuch a diſtance from the middle Term, as will be expreſſed by $\frac{1}{4}\sqrt{2n}$ very near; ſo in the Caſe above mentioned, wherein n was ſuppoſed $= 3600$, $\frac{1}{4}\sqrt{2n}$ will be about 21.2 nearly, which in reſpect to 3600, is not above $\frac{1}{169}$th part: ſo that it is an equal Chance nearly, or rather ſomething more, that in 3600 Experiments, in each of which an Event may as well happen as fail, the Exceſs of the happenings or failings above 1800 times will be no more than about 21.

COROLLARY 6.

If l be interpreted by $\sqrt{n}$, the Series will not converge ſo faſt as it did in the former Caſe when l was interpreted by $\frac{1}{2}\sqrt{n}$, for here no leſs than 12 or 13 Terms of the Series will afford a tolerable approximation, and it would ſtill require more Terms, according as l bears a greater proportion to $\sqrt{n}$: for which reaſon I make uſe in this Caſe of the Artifice of Mechanic Quadratures, firſt invented by Sir *Iſaac Newton*, and ſince proſecuted by Mr. *Cotes*, Mr. *James Stirling*, myſelf, and perhaps others; it conſiſts in determining the Area of a Curve nearly, from knowing a certain number of its Ordinates A, B, C, D, E, F, &c. placed at equal Intervals, the more Ordinates there are, the more exact will the Quadrature be; but here I confine myſelf to four, as being ſufficient for my purpoſe: let us therefore ſuppoſe that the four Ordinates are A, B, C, D, and that the Diſtance between the firſt and laſt is denoted by l, then

l, then the Area contained between the firſt and the laſt will be $\frac{1\times\overline{A+D}+3\times\overline{B+C}}{8}\times l$; now let us take the Diſtances $0\sqrt{n}$, $\frac{1}{6}\sqrt{n}$, $\frac{2}{6}\sqrt{n}$, $\frac{3}{6}\sqrt{n}$, $\frac{4}{6}\sqrt{n}$, $\frac{5}{6}\sqrt{n}$, $\frac{6}{6}\sqrt{n}$, of which every one exceeds the preceding by $\frac{1}{6}\sqrt{n}$, and of which the laſt is $\sqrt{n}$; of theſe let us take the four laſt, *viz.* $\frac{3}{6}\sqrt{n}$, $\frac{4}{6}\sqrt{n}$, $\frac{5}{6}\sqrt{n}$, $\frac{6}{6}\sqrt{n}$, then taking their Squares, doubling each of them, dividing them all by n, and prefixing to them all the Sign $-$, we ſhall have $-\frac{1}{2}$, $-\frac{8}{9}$, $-\frac{25}{18}$, $-\frac{2}{1}$, which muſt be looked upon as Hyperbolic Logarithms, of which conſequently the correſponding numbers, *viz.* 0.60653, 0.41111, 0.24935, 0.13534 will ſtand for the four Ordinates A, B, C, D. Now having interpreted l by $\frac{1}{2}\sqrt{n}$, the Area will be found to be $= 0.170203\times\sqrt{n}$, the double of which being multiplied by $\frac{2}{\sqrt{nc}}$, the product will be 0.27160; let therefore this be added to the Area found before, that is, to 0.682688, and the Sum 0.95428 will ſhew what, after a number of Trials denoted by n, the Probability will be of the Event's neither happening oftner than $\frac{1}{2}n+\sqrt{n}$ times, nor more rarely than $\frac{1}{2}n-\sqrt{n}$, and therefore the Probability of the contrary will be 0.04572: which ſhews that the Odds of the Event's neither happening oftner nor more rarely than within the Limits aſſigned are 21 to 1 nearly.

And by the ſame way of reaſoning, it will be found that the Probability of the Event's neither appearing oftner than $\frac{1}{2}n+\frac{3}{2}\sqrt{n}$, nor more rarely than $\frac{1}{2}n-\frac{3}{2}\sqrt{n}$ will be 0.99874, which will make it that the Odds in this Caſe will be 369 to 1 nearly.

To apply this to particular Examples, it will be neceſſary to eſtimate the frequency of an Event's happening or failing by the Square-root of the number which denotes how many Experiments have been, or are deſigned to be taken; and this Square-root, according as it has been already hinted at in the fourth Corollary, will be as it were the *Modulus* by which we are to regulate our Eſtimation; and therefore ſuppoſe the number of Experiments to be taken is 3600, and that it were required to aſſign the Probability of the Event's neither happening oftner than 2850 times, nor more rarely than 1750, which two numbers may be varied at pleaſure, provided they be equally diſtant from the middle Sum 1800, then make the half difference between the two numbers 1850 and 1750, that is, in this Caſe, $50 = ſ\sqrt{n}$; now having ſuppoſed $3600 = n$, then $\sqrt{n}$ will be $= 60$, which will make it that 50 will be $= 60ſ$, and conſequently $ſ = \frac{50}{60} = \frac{5}{6}$; and therefore if we take the proportion, which in an infinite power, the double Sum of the Terms correſponding to the Interval $\frac{5}{6}\sqrt{n}$, bears to the Sum of all the Terms, we ſhall have the Probability required exceeding near.

LEMMA 2.

In any Power $\overline{a+b}^{\,n}$ expanded, the greateſt Term is that in which the Indices of the Powers of a and b, have the ſame proportion to one another as the Quantities themſelves a and b; thus taking the 10th Power of $a+b$, which is $a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45a^2b^6 + 10ab^7 + b^{10}$; and ſuppoſing that the proportion of a to b is as 3 to 2, then the Term $210a^6b^4$ will be the greateſt, by reaſon that the Indices of the Powers of a and b, which are in that Term, are in the proportion of 3 to 2; but ſuppoſing the proportion of a to b had been as 4 to 1, then the Term $45a^8b^2$ had been the greateſt.

LEMMA 3.

If an Event ſo depends on Chance, as that the Probabilities of its happening or failing be in any aſſigned proportion, ſuch as may be ſuppoſed of a to b, and a certain number of Experiments be deſigned to be taken, in order to obſerve how often the Event will happen or fail; then the Probability that it ſhall neither happen more frequently than ſo many times as are denoted by $\frac{an}{a+b} + l$, nor more rarely than ſo many times as are denoted by $\frac{an}{a+b} - l$, will be found as follows:

Let L and R be equally diſtant by the Interval l from the greateſt Term; let alſo S be the Sum of the Terms included between L and R, together with thoſe Extreams, then the Probability required will be rightly expreſſed by $\frac{S}{\overline{a+b}^{\,n}}$.

COROLLARY 8.

The Ratio which, in an infinite Power denoted by n, the greateſt Term bears to the Sum of all the reſt, will be rightly expreſſed by the Fraction $\frac{a+b}{\sqrt{abnc}}$, wherein c denotes, as before, the Circumference of a Circle for a Radius equal to Unity.

COROLLARY 9.

If, in an infinite Power, any Term be diſtant from the Greateſt by the Interval l, then the Hyperbolic Logarithm of the Ratio which that Term bears to the Greateſt will be expreſſed by the Fraction $-\frac{\overline{a+b}^2}{2abn} \times ll$; provided the Ratio of l to n be not a finite Ratio, but ſuch a one as may be conceived between any given number p and $\sqrt{n}$, ſo that l be expreſſible by $p\sqrt{n}$, in which Caſe the two Terms L and R will be equal.

COROLLARY 10.

If the Probabilities of happening and failing be in any given Ratio of inequality, the Problems relating to the Sum of the Terms of the Binomial $\overline{a+b}^n$ will be ſolved with the ſame facility as thoſe in which the Probabilities of happening and failing are in a Ratio of Equality.

REMARK I.

From what has been ſaid, it follows, that Chance very little diſturbs the Events which in their natural Inſtitution were deſigned to happen or fail, according to ſome determinate Law; for if in order to help our conception, we imagine a round piece of Metal, with two poliſhed oppoſite faces, differing in nothing but their colour, whereof one may be ſuppoſed to be white, and the other black; it is plain that we may ſay, that this piece may with equal facility exhibit a white or black face, and we may even ſuppoſe that it was framed with that particular view of ſhewing ſometimes one face, ſometimes the other, and that conſequently if it be toſſed up Chance ſhall decide the appearance; But we have ſeen in our LXXII^d Problem, that altho' Chance may produce an inequality of appearance, and ſtill a greater inequality according to the length of time in which it may exert itſelf, yet the appearances, either one way or the other, will perpetually tend to a proportion of Equality: But beſides, we have ſeen in the preſent Problem, that in a great number of Experiments, ſuch as 3600, it would be the Odds of above 2 to 1, that one of the Faces, ſuppoſe the white, ſhall not appear more frequently than 1830 times, nor more rarely than 1770, or in other Terms, that it ſhall not be above or under the perfect Equality by more than $\frac{1}{120}$ part of the whole number of appearances; and by the ſame Rule, that if the number of Trials had been 14400 inſtead of 3600, then ſtill it would be above the Odds of 2 to 1, that the appearances either one way or other would not deviate from perfect Equality by more than $\frac{1}{260}$ part of the whole: and in 1000000 Trials it would be the Odds of above 2 to 1, that the deviation from perfect Equality would not be more than by $\frac{1}{2000}$ part of the whole. But the Odds would increaſe at a prodigious rate, if inſtead of taking ſuch narrow limits on both ſides the Term of Equality, as are repreſented by $\frac{1}{2}\sqrt{n}$, we double thoſe Limits or triple them; for in the firſt Caſe the Odds would become 21 to 1, and in the ſecond 369 to 1, and ſtill be vaſtly greater if we were to quadruple them, and at laſt be infinitely great; and yet whether we double, triple or quadruple them, *&c.* the Extenſion of thoſe Limits will bear but an inconſiderable proportion to the whole, and none at all, if the whole be infinite; of which the reaſon will eaſily be perceived by Mathematicians, who know, that the Square-root of any Power bears ſo much a leſs proportion to that Power, as the Index of it is great.

What we have ſaid is alſo applicable to a Ratio of Inequality, as appears from our 9th Corollary. And thus in all Caſes it will be found, that *altho' Chance produces Irregularities, ſtill the Odds will be infinitely great, that in proceſs of Time, thoſe Irregularities will bear no proportion to the recurrency of that Order which naturally reſults from* ORIGINAL DESIGN.

REMARK II.

As, upon the Suppoſition of a certain determinate Law according to which any Event is to happen, we demonſtrate that the Ratio of Happenings will continually approach to that Law, as the Experiments or Obſervations are multiplied: ſo, *converſely*, if from numberleſs Obſervations we find the Ratio of the Events to converge to a determinate quantity, as to the Ratio of P to Q; then we conclude that this Ratio expreſſes the determinate Law according to which the Event is to happen.

For let that Law be expreſſed not by the Ratio P : Q, but by ſome other, as R : S; then would the Ratio of the Events converge to this laſt, not to the former: which contradicts our *Hypotheſis.* And the like, or greater, Abſurdity follows, if we ſhould ſuppoſe the

Event not to happen according to any Law, but in a manner altogether deſultory and uncertain; for then the Events would converge to no fixt Ratio at all.

Again, as it is thus demonſtrable that there are, in the conſtitution of things, certain Laws according to which Events happen, it is no leſs evident from Obſervation, that thoſe Laws ſerve to wiſe, uſeful and beneficent purpoſes; to preſerve the ſtedfaſt Order of the Univerſe, to propagate the ſeveral Species of Beings, and furniſh to the ſentient Kind ſuch degrees of happineſs as are ſuited to their State.

But ſuch Laws, as well as the original Deſign and Purpoſe of their Eſtabliſhment, muſt all be *from without*; the *Inertia* of matter, and the nature of all created Beings, rendering it impoſſible that any thing ſhould modify its own eſſence, or give to itſelf, or to any thing elſe, an original determination or propenſity. And hence, if we blind not ourſelves with metaphyſical duſt, we ſhall be led, by a ſhort and obvious way, to the acknowledgment of the great MAKER and GOVERNOUR of all; *Himſelf all-wiſe, all-powerful* and *good*.

Mr. *Nicolas Bernoulli* *, a very learned and good Man, by not connecting the latter part of our reaſoning with the firſt, was led to diſcard and even to vilify this Argument from *final Cauſes*, ſo much inſiſted on by our beſt Writers; particularly in the Inſtance of the nearly equal numbers of *male* and *female* Births, adduced by that excellent Perſon the late Dr. *Arbuthnot*, in *Phil. Tranſ.* N°. 328.

Mr. *Bernoulli* collects from Tables of Obſervations continued for 82 years, that is from *A. D.* 1629 to 1711, that the number of Births in *London* was, at a *medium*, about 14000 yearly: and likewiſe, that the number of *Males* to that of *Females*, or the facility of their production, is nearly as 18 to 17. But he thinks it the greateſt weakneſs to draw any Argument from this againſt the Influence of *Chance* in the production of the two Sexes. For, ſays he,

" Let 14000 Dice, each having 35 faces, 18 white and 17 black,
" be thrown up, and it is great Odds that the numbers of white and
" black faces ſhall come as near, or nearer, to each other, as the
" numbers of Boys and Girls do in the Tables."

To which the ſhort anſwer is this: Dr. *Arbuthnot* never ſaid, " that ſuppoſing the facility of the production of a Male to that

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* See his two Letters to Mr. *de Monmort*, one dated at *London*, 11 *Oct.* 1712, the other from *Paris*, 23 *Jan.* 1713, in the Appendix to the *Analyſe des Jeux de hazard*, 2d Edit.

" of the production of a female to be already *fixt* to nearly the Ratio " of equality, or to that of 18 to 17; he was *amazed* that the Ratio " of the numbers of Males and Females born ſhould, for many years, " keep within ſuch narrow bounds:" the only Propoſition againſt which Mr. *Bernoulli*'s reaſoning has any force.

But he might have ſaid, and we do ſtill inſiſt, that " as, from " the Obſervations, we can, with Mr. *Bernoulli*, infer the facilities of production of the two Sexes to be nearly in a Ratio of " equality; ſo from this Ratio once diſcovered, and *manifeſtly ſerving to a wiſe purpoſe*, we conclude the Ratio itſelf, or if you will " the *Form of the Die*, to be an Effect of *Intelligence* and *Deſign*." As if we were ſhewn a number of Dice, each with 18 white and 17 black faces, which is Mr. *Bernoulli*'s ſuppoſition, we ſhould not doubt but that thoſe Dice had been made by ſome Artiſt; and that their form was not owing to *Chance*, but was adapted to the particular purpoſe he had in View.

Thus much was neceſſary to take off any impreſſion that the authority of ſo great a name might make to the prejudice of our argument. Which, after all, being level to the loweſt underſtanding, and falling in with the common ſenſe of mankind, needed no formal Demonſtration, but for the ſcholaſtic ſubtleties with which it may be perplexed; and for the abuſe of certain words and phraſes; which ſometimes are imagined to have a meaning merely becauſe they are often uttered.

Chance, as we underſtand it, ſuppoſes the *Exiſtence* of things, and their general known *Properties:* that a number of Dice, for inſtance, being thrown, each of them ſhall ſettle upon one or other of its Baſes. After which, the *Probability* of an aſſigned Chance, that is of ſome particular diſpoſition of the Dice, becomes as proper a ſubject of Inveſtigation as any other quantity or Ratio can be.

But *Chance*, in atheiſtical writings or diſcourſe, is a ſound utterly inſignificant: It imports no determination to any *mode of Exiſtence*; nor indeed to *Exiſtence* itſelf, more than to *non-exiſtence*; it can neither be defined nor underſtood: nor can any Propoſition concerning it be either affirmed or denied, excepting this one, " That " it is a mere word."

The like may be ſaid of ſome other words in frequent uſe; as *fate*, *neceſſity*, *nature*, a *courſe of nature* in contradiſtinction to the *Divine energy:* all which, as uſed on certain occaſions, are mere ſounds: and yet, by artful management, they ſerve to found ſpecious concluſions: which however, as ſoon as the latent fallacy of the *Term* is detected, appear to be no leſs abſurd in themſelves, than they commonly are hurtful to ſociety. I ſhall

I ſhall only add, That this method of Reaſoning may be uſefully applied in ſome other very intereſting Enquiries; if not to force the Aſſent of others by a ſtrict Demonſtration, at leaſt to the Satisfaction of the Enquirer himſelf: and ſhall conclude this Remark with a paſſage from the *Ars Conjectandi* of Mr. *James Bernoulli*, *Part* IV. *Cap.* 4. where that acute and judicious Writer thus introduceth his Solution of the Problem for *Aſſigning the Limits within which, by the repetition of Experiments, the Probability of an Event may approach indefinitely to a Probability given*, "Hoc igitur eſt illud Problema &c." *This, ſays he, is the Problem which I am now to impart to the Publick, after having kept it by me for twenty years: new it is, and difficult, but of ſuch excellent uſe, that it gives a high value and dignity to every other Branch of this Doctrine.* Yet there are Writers, of a Claſs indeed very different from that of *James Bernoulli*, who inſinuate as if the *Doctrine of Probabilities* could have no place in any ſerious Enquiry; and that Studies of this kind, trivial and eaſy as they be, rather diſqualify a man for reaſoning on every other ſubject. Let the Reader chuſe.

PROBLEM LXXIV.

To find the Probability of throwing a Chance aſſigned a given number of times without intermiſſion, in any given number of Trials.

SOLUTION.

Let the Probability of throwing the Chance in any one Trial be repreſented by $\frac{a}{a+b}$, and the Probability of the contrary by $\frac{b}{a+b}$: Suppoſe n to repreſent the number of Trials given, and p the number of times that the Chance is to come up without intermiſſion; then ſuppoſing $\frac{b}{a+b} = x$, take the quotient of Unity divided by $1 - x - axx - aax^3 - a^3x^4 - a^4x^5 \text{ - - - - - - - - - - } - a^{p-1}x^p$, and having taken as many Terms of the Series reſulting from that diviſion, as there are Units in $n - p + 1$, multiply the Sum of the whole by $\frac{a^p x^p}{b^p}$, or by $\frac{a^p}{\overline{a+b}|^p}$, and that Product will expreſs the Probability required.

EXAMPLE I.

Let it be required to throw the Chance aſſigned three times together, in 10 trials, when a and b are in a ratio of Equality, otherwiſe when each of them is equal to Unity; then having divided 1 by $1-x-xx-x^3$, the Quotient continued to ſo many Terms as there are Units in $n-p+1$, that is, in this Caſe to $10-3+1=8$, will be $1+x+2xx+4x^3+7x^4+13x^5+24x^6+44x^7$. Where x being interpreted by $\frac{b}{a+b}$, that is in this Caſe by $\frac{1}{2}$, the Series will become $1+\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+\frac{7}{16}+\frac{13}{32}+\frac{24}{64}+\frac{44}{128}$, of which the Sum is $\frac{520}{128}=\frac{65}{16}$, and this being multiplied by $\frac{a^p x^p}{b^p}$, that is, in this Caſe by $\frac{1}{8}$, the Product will be $\frac{65}{128}$, and therefore 'tis ſomething more than an equal Chance, that the Chance aſſigned will be thrown three times together ſome time in 10 Trials, the Odds for it being 65 to 63.

N. B. The continuation of the Terms of thoſe Series is very eaſy; for in the Caſe of the preſent Problem, the Coefficient of any Term is the Sum of 3 of the preceding; and in all Caſes, 'tis the Sum of ſo many of the preceding Coefficients as are denoted by the number p.

But if, in the foregoing Example, the ratio of a to b was of inequality, ſuch as, for inſtance 2 to 1, then according to the preſcription given before, divide Unity by $1-x-2xx-4x^3$, and the Quotient will be $1+x+3xx+9x^3+19x^4+49x^5+123x^6+297x^7$, in which the quantity x, which has univerſally been ſuppoſed $=\frac{b}{a+b}$, will in this Caſe be $=\frac{1}{3}$; wherefore in the preceding Series having interpreted x by $\frac{1}{3}$, we ſhall find the Sum of 8 of its Terms will be $=\frac{5904}{2187}=\frac{74}{27}$, and this being multiplied by $\frac{a^p x^p}{b^p}$ which in this Caſe is $\frac{8}{27}$, the Product $\frac{592}{729}$ will expreſs the Probability required, ſo that there are the Odds of 592 to 137, that the Chance aſſigned will happen three times together in 10 Trials or before; and only the Odds of 41 to 40 that it does not happen three times together in 5.

After having given the general Rule, it is proper to conſider of Expedients to make the Calculation more eaſy; but before we proceed, it is proper to take a new Caſe of this Problem: Suppoſe therefore it be required to find the Probability of throwing the Chance aſſigned 4 times together in 21 Trials. And firſt let us ſuppoſe the Chance aſſigned to be of Equality, then we ſhould begin to divide Unity by $1 - x - xx - x^3 - x^4$; but if we conſider that the Terms $x + xx + x^3 + x^4$ are in geometric Progreſſion, and that the Sum of that Progreſſion is $\frac{x - x^5}{1 - x}$, if we ſubtract that from 1, the remainder $\frac{1 - 2x + x^5}{1 - x}$ will be equivalent to $1 - x - xx - x^3 - x^4$, and conſequently $\frac{1 - x}{1 - 2x + x^5}$ will be equivalent to the Quotient of Unity divided by $1 - x - xx - x^3 - x^4$; and therefore by that expedient, the moſt complex Caſe of this Problem will be reduced to the contemplation of a Trinomial; let us therefore begin to take ſo many Terms of the Series reſulting from the Diviſion of Unity by the Trinomial $1 - 2x + x^5$ as there are Units in $n - p + 1$, that is in $21 - 4 + 1$, or 18, and thoſe Terms will be $1 + 2x + 4x^2 + 8x^3 + 16x^4 + 31x^5 + 60x^6 + 116x^7 + 224x^8 + 432x^9 + 833x^{10} + 1606x^{11} + 3096x^{12} + 5968x^{13} + 11494x^{14} + 22155x^{15} + 42704x^{16} + 82312x^{17}$. Now although theſe Terms may ſeem at firſt ſight to be acquired by very great labour, yet if we conſider what has been explained before concerning the nature of a recurring Series, we ſhall find that each Coefficient of the Series is generated from the double of the laſt, ſubtracting once the Coefficient of that Term which ſtands 5 places from the laſt incluſive; ſo that for inſtance if we wanted one Term more, conſidering that the laſt Coefficient is 82312, and that the Coefficient of that Term which ſtands five places from the laſt incluſive is 5968, then the Coefficient required will be twice 82312, wanting once 5968, which will make it 158656, ſo that the Term following the laſt will be $158656x^{18}$.

But to make this more conſpicuous, if we take the Binomial $2x - x^5$, and raiſe it ſucceſſively to the Powers, whoſe Indices are 0, 1, 2, 3, 4, 5, 6. &c. and add all thoſe powers together, and write againſt one another all the Terms which have the ſame power of x, we ſhall have a very clear view of the quotient of 1 divided by $1 - 2x + x^5$. Now it will ſtand thus, ſuppoſing that a ſtands for 2.

$$\begin{array}{llll}
1 & & & \\
+ax & & & \\
+a^2xx & & & \\
+a^3x^3 & & & \\
+a^4x^4 & & & \\
+a^5x^5 & -x^5 & & \\
+a^6x^6 & -2ax^6 & & \\
+a^7x^7 & -3aax^7 & & \\
+a^8x^8 & -4a^3x^8 & & \\
+a^9x^9 & -5a^4x^9 & & \\
+a^{10}x^{10} & -6a^5x^{10} & +1x^{10} & \\
+a^{11}x^{11} & -7a^6x^{11} & +3ax^{11} & \\
+a^{12}x^{12} & -8a^7x^{12} & +6aax^{12} & \\
+a^{13}x^{13} & -9a^8x^{13} & +10a^3x^{13} & \\
+a^{14}x^{14} & -10a^9x^{14} & +15a^4x^{14} & \\
+a^{15}x^{15} & -11a^{10}x^{15} & +21a^5x^{15} & -1x^{15} \\
+a^{16}x^{16} & -12a^{11}x^{16} & +28a^6x^{16} & -4ax^{16} \\
+a^{17}x^{17} & -13a^{12}x^{17} & +36a^7x^{17} & -10aax^{17}
\end{array}$$

When the Terms have been diſpoſed in that manner, it will be eaſy to ſum them up by the help of a Theorem which may be ſeen pag. 224. Now a being $= 2$, and $x = \frac{1}{2}$, every one of the Terms of the firſt Column will be equal to 1, and therefore the Sum of the firſt Column is ſo many Units as there are Terms, which Sum conſequently will be 18; but the Terms of the ſecond Column being reduced to their proper Value, will conſtitute the Series $\frac{1}{32} + \frac{2}{32} + \frac{3}{32} + \frac{4}{32} + \frac{5}{32} + \frac{6}{32} + \frac{7}{32} + \frac{8}{32} + \frac{9}{32} + \frac{10}{32} + \frac{11}{32} + \frac{12}{32} + \frac{13}{32}$ of which the Sum will be $\frac{91}{32}$; the Terms of the third Column will conſtitute the Series $\frac{1}{1024} + \frac{3}{1024} + \frac{6}{1024} + \frac{10}{1024} + \frac{15}{1024} + \frac{21}{1024} + \frac{28}{1024} + \frac{36}{1024}$ of which the Sum is $\frac{120}{1024}$; the Terms of the fourth Column added together are $\frac{15}{32768}$, and therefore the Sum of all Terms may be expreſſed by $18 - \frac{91}{32} + \frac{120}{1024} - \frac{15}{32768} = \frac{500465}{32768}$.

But this Sum ought to be multiplied by $1 - x$, that is, by $1 - \frac{1}{2} = \frac{1}{2}$, which will make the Product to be $\frac{500465}{65536}$.

Neverthelefs, this Multiplication by $1 - x$, takes off too much from the true Sum, by one half of the loweſt Term of each Column, therefore that half muſt be added to the foregoing Sum; now all the loweſt Terms of each Column put together will be $1 - \frac{13}{32} + \frac{36}{1024} - \frac{10}{32768} = \frac{20598}{32798}$, of which the half $\frac{10299}{32768}$ ought to be added to the Sum $\frac{500465}{65536}$, which will make the true Sum to be $\frac{521063}{65536}$; but this is farther to be multiplied by $\frac{a^p x^p}{b^p}$, which by reaſon that a and b are in a ratio of equality will be reduced to $x^p = \frac{1}{16}$; and therefore the Sum $\frac{521063}{65536}$ ought to be divided by 16, which will make it to be $\frac{521063}{1048576}$: and this laſt Fraction will denote the Probability of producing the Chance aſſigned 4 times ſucceſſively ſome time in 21 Trials, the Odds againſt it being 527513 to 521063, which is about 82 to 81.

But what is remarkable in this Problem is this, that the oftner the Chance aſſigned is to be produced ſucceſſively, the fewer Columns will be neceſſary to be uſed to have a ſufficient Approximation, and in all high Caſes, it will be ſufficient to uſe only the firſt and ſecond, or three at moſt, whereof the firſt is a geometric Progreſſion, of which a very great number of Terms will be as eaſily ſummed up, as a very ſmall number; and the ſecond Column by what we have ſaid concerning the nature of a recurring Series, as eaſily as the firſt, and in ſhort all the Columns.

But now 'tis time to conſider the Caſe wherein a to b has a ratio of inequality; we had ſaid before that in this Caſe we ought to divide Unity by $1 - x - axx - aax^3 - a^3x^4 \text{------} - a^{p-1}x^p$, but all the Terms after the firſt which is 1, conſtitute a geometric Progreſſion, of which the firſt is x, and the laſt $a^{p-1}x^p$, and therefore the Sum of that Progreſſion is $\frac{x - a^p x^{p+1}}{1 - ax}$, and this being ſubtracted from Unity, the remainder will be

$$\frac{1 - ax + a^p x^{p+1} - x}{1 - ax},$$

and therefore Unity being divided by the Series above-written will be

$\dfrac{1-ax}{\begin{array}{l}1-ax+a^{p}x^{p+1}\\ \;\;-\;x\end{array}}$, and if $a+1$ be ſuppoſed $=m$, this Fraction will be $\frac{1-ax}{1-mx-a^{p}x^{p+1}}$, and therefore if we raiſe ſucceſſively $mx-a^{p}x^{p+1}$ to the ſeveral Powers denoted by 0, 1, 2, 3, 4, 5, 6, &c. and rank all thoſe Powers in ſeveral Columns, and write againſt one another all the Terms that have the ſame power of x, we ſhall be able to ſum up every Column extended to the number of Terms denoted by $n-p+1$, which being done, the whole muſt be multiplied by $1-ax$, and to the Sum is to be added the Sum of the loweſt Term of each Column multiplied by ax.

But if it be required to aſſign what number of Games are neceſſary, in all Caſes, to make it an equal Chance whether or not p Games will be won without intermiſſion, it may be done by approximation, thus; let $\frac{\overline{a+b}\vert^{p}-a^{p}}{a^{p}}$ be ſuppoſed $=q$, and let $\frac{a+b}{b}-\frac{a^{p}}{b\times\overline{a+b}\vert^{p-1}}$ be ſuppoſed $=r$, then the number of Games required will be expreſſed by $\frac{7}{10}qr$; thus ſuppoſing $a=1$, $b=1$, $p=6$, then the number of Games would be found between 86 and 87; but if a be ſuppoſed $=1$, and $b=2$, ſtill ſuppoſing $p=6$, the number of Games requiſite to that effect would be found to be between 763 and 764; but it is to be obſerved, that the greater the number p is, ſo much the more exact will the Solution prove.

A

TREATISE

OF

ANNUITIES

ON

LIVES:

Dedicated to

The RIGHT HONOURABLE

GEORGE Earl of *MACCLESFIELD*

PRESIDENT of the ROYAL SOCIETY.

PREFACE

TO THE

Second EDITION.

DR. Halley *publiſhed in the* Philoſophical Tranſactions, *N*°. 196. *an Eſſay concerning the Valuation of Lives*; *it was partly built upon five Years Obſervation of the Bills of Mortality taken at* Breſlaw, *the Capital of* Sileſia, *and partly on his own Calculation.*

Altho' he had thereby confirmed the great Opinion which the World entertained of his Skill and Sagacity, yet he was ſenſible, that his Tables and Calculations were ſuſceptible of farther Improvements; of this he expreſſed his Senſe in the following Words; Were this Calculus founded on the Experience of a very great Number of Years, it would very well be worth the while, to think of Methods to facilitate the Computation of two, three or more Lives.

From whence it appears, that the Table of Obſervations being only the Reſult of a few Years Experience, it was not ſo entirely to be depended upon, as to make it the Foundation of a fixed and unalterable Valuation of Annuities on Lives; and that even admitting ſuch a Table could be obtained, as might be grounded on the Experience of a great Number of Years, ſtill the Method of applying it to the Valuation of ſeveral Lives, would be extremely laborious, conſidering the vaſt Number of Operations, that would be requiſite to combine every Year of each Life with every Year of all the other Lives.

The Subject of Annuities on Lives, had been long neglected by me, partly prevented by other Studies, partly wanting the neceſſary means to treat of it as it deſerved: But two or three Years after the Publication of the firſt Edition of my Doctrine of Chances, *I took the Subject into Conſideration; and conſulting Dr.* Halley's *Table of Obſervations, I found that the Decrements of Life, for conſiderable Intervals of Time, were in Arithmetic Progreſſion; for Inſtance, out of* 646 *Perſons of twelve Years of Age, there remain* 640 *after one Year;* 634 *after two Years;* 628, 622, 616, 610, 604, 598, 592, 586, *after* 3, 4, 5, 6, 7, 8, 9, 10 *Years reſpectively, the common Difference of thoſe Numbers being* 6.

Examining afterwards other Caſes, I found that the Decrements of Life for ſeveral Years were ſtill in Arithmetic Progreſſion; which may be obſerved from the Age of 54, *to the Age of* 71, *where the Difference for* 17 *Years together, is conſtantly* 10.

After having thoroughly examined the Tables of Obſervation, and diſcovered that Property of the Decrements of Life, I was inclined to compoſe a Table of the Values of Annuities on Lives, by keeping cloſe to the Tables of Obſervation; which would have been done with Eaſe, by taking in the whole Extent of Life, ſeveral Intervals whether equal or unequal: However, before I undertook the Taſk, I tried what would

would be the Reſult, of ſuppoſing thoſe Decrements uniform from the Age of Twelve; being ſatisfied that the Exceſſes ariſing on one ſide, would be nearly compenſated by the Defects on the other; then comparing my Calculation with that of Dr. Halley, *I found the Concluſion ſo very little different, that I thought it ſuperfluous to join together ſeveral different Rules, in order to compoſe a ſingle one: I need not take notice, that from the Time of Birth to the Age of Twelve, the Probabilities of Life increaſe, rather than decreaſe, which is a Reaſon of the apparent Irregularity of the Tables in the beginning.*

Another thing was neceſſary to my Calculation, which was, to ſuppoſe the Extent of Life confined to a certain Period of Time, which I ſuppoſe to be at 86: *What induced me to aſſume that Suppoſition was* 1*ſt, That Dr.* Halley *terminates his Tables of Obſervations at the* 84th *Year; for altho' out of* 1000 *Children of one Year of Age, there are twenty, who, according to Dr.* Halley's *Tables, attain to the Age of* 84 *Years, this Number of* 20 *is inconſiderable, and would ſtill have been reduced, if the Obſervations had been carried two Years farther.* 2°. *It appears from the Tables of* Graunt, *who printed the firſt Edition of his Book above* 80 *Years ago, that out of* 100 *new-born Children, there remained not one after* 86 *Years; this was deduced from the Obſervations of ſeveral Years, both in the City and the Country, at a Time when the City being leſs populous, there was a greater Facility of coming at the Truth, than at preſent.* 3°. *I was farther confirmed in my Hypotheſis, by Tables of Obſervation made in* Switzerland, *about the Beginning of this Century, wherein the Limit of Life is placed at* 86: *As for what is alledged, that by ſome Obſervations of late Years, it appears, that Life is carried to* 90, 95, *and even to* 100 *Years; I am no more moved by it, than by the Examples of* Parr *or* Jenkins, *the firſt of whom lived* 152 *Years, and the other* 167. *To this may be added, that the Age for purchaſing Annuities for Life, ſeldom exceeds* 70, *at which Term, Dr.* Halley *ends his Tables of the Valuation of Lives.*

The greateſt Difficulty that occurred to me in this Speculation, was to invent practical Rules that might eaſily be applied to the Valuation of ſeveral Lives; which, however, was happily overcome, the Rules being ſo eaſy, that by the Help of them, more can be performed in a Quarter of an Hour, than by any Method before extant, in a Quarter of a Year.

Since the Publication of my firſt Edition, which was in 1724, *I made ſome Improvements to it, as may be ſeen in the ſecond Edition of my* Doctrine of Chances; *but this Edition of the Annuities has many Advantages over the former, and that in reſpect to the Diſpoſition of the Precepts, the Conciſeneſs of the Rules, the Multiplicity of Problems, and Uſefulneſs of the Tables I have invented.*

Before I make an End of this Preface, I think it proper to obſerve, that altho' I have given Rules for finding the Value of Annuities for any Rate of Intereſt, yet I have confined myſelf in my Tables, to the ſeveral Rates of 4, 5 *and* 6 per Cent. *which may be interpreted, as if I thought it reaſonable, that when Land ſcarce produces three and a half* per Cent. *and South-Sea Annuities barely that Intereſt, yet the Purchaſer of an Annuity ſhould make* 4 per Cent. *or above; but thoſe Caſes can hardly admit of Compariſon, it being well known, that Land in Fee-ſimple procures to the Proprietor Credit, Honour, Reputation, and other Advantages, in conſideration of which, he is contented with a ſmaller Income. As to the Value of South-Sea Annuities, it has its Foundation on the Punctuality of Payments, and on a Parliamentary Security; but Annuities on Lives, have not the former Security, and ſeldom the latter.*

It was found neceſſary, however, in a ſubſequent Edition, to add the Tables of 3 and 3½ *per Cent.* Intereſt.

Of ANNUITIES on LIVES.

Part I. containing the Rules and Examples.

BEFORE I come to the Solution of Queſtions on Lives, it will be neceſſary to explain the Meaning of ſome Words which I ſhall often have occaſion to mention.

1°. Suppoſing the *Probabilities of Life* to decreaſe in Arithmetic Progreſſion in ſuch manner, as that ſuppoſing, for Inſtance, 36 Perſons each of the Age of 50, if after one Year expired there remain but 35, after two 34, after three 33, and ſo on; it is very plain that ſuch Lives would neceſſarily be extinct in 36 Years, and that therefore the *Probabilities of living* 1, 2, 3, 4, 5, *&c.* Years from this Age of 50 would fitly be repreſented by the Fractions $\frac{35}{36}$, $\frac{34}{36}$, $\frac{33}{36}$, $\frac{32}{36}$, $\frac{31}{36}$, *&c.* which decreaſe in Arithmetic Progreſſion.

I will not ſay that the Decrements of Life are preciſely in that Proportion; ſtill comparing that Hypotheſis with the Table of Dr. *Halley*, from the Obſervations made at *Breſlaw*, they will be found to be exceedingly approaching.

2°. I call that the *Complement of Life*, which remains from the Age given, to the Time of the Extinction of Life, which will be at 86, according to our Hypotheſis. Thus ſuppoſing an Age of 50, becauſe the Difference between 50 and 86 is 36, I call 36 the *Complement of Life*.

3°. I call that the *Rate of Intereſt* which is properly the Amount of one Pound, put out at Intereſt for one Year; otherwiſe one Pound joined with the Intereſt it produces in one Year: thus ſuppoſing Intereſt at 5 *per Cent* the Intereſt of 1 *l.* would be 0.05, which being joined to the Principal 1, produces 1.05; which is what I call the *Rate of Intereſt*.

PROBLEM I.

Suppoſing the Probabilities of Life to decreaſe in Arithmetic Progreſſion, to find the Value of an Annuity upon a Life of an Age given.

SOLUTION.

Let the Rent or Annuity be ſuppoſed $=1$, the Rate of Intereſt $=r$, the Complement of Life $=n$, the Value of an Annuity certain to continue during n Years $= P$, then will the Value of the Life be $\frac{1 - \frac{r}{n}P}{r - 1}$, which is thus expreſſed in Words at length;

Take the Value of an Annuity certain for ſo many Years, as are denoted by the Complement of Life; multiply this Value by the Rate of Intereſt, and divide the Product by the Complement of Life, then let the Quotient be ſubtracted from Unity, and let the Remainder be divided by the Intereſt of 1 l. *then this laſt Quotient will expreſs the Value of an Annuity for the Age given.*

Thus ſuppoſe it were required to find the preſent Value of an Annuity of 1 *l.* for an Age of 50, Intereſt being at 5 *per Cent.*

The Complement of Life being 36, let the Value of an Annuity certain, according to the given Rate of Intereſt, be taken out of the Tables annexed to this Book, this Value will be found to be 16.5468.

Let this Value be multiplied by the Rate of Intereſt 1.05, the Product will be 17.3741.

Let this Product be divided by the Complement of Life, *viz.* by 36, the Quotient will be 0.4826.

Subtract this Quotient from Unity, the Remainder will be 0.5174.

Laſtly, divide this Quotient by the Intereſt of 1 *l.* *viz.* by 0.05, and the new Quotient will be 10.35; which will expreſs the Value of an Annuity of 1 *l.* or how many Years Purchaſe the ſaid Life of 50 is worth.

And in the ſame manner, if Intereſt of Money was at 6 *per Cent.* an Annuity upon an Age of 50, would be found worth 9.49 Years Purchaſe.

But as I have annexed to this Treatiſe the Values of Annuities for an Intereſt of 3, $3\frac{1}{2}$, 4, 5, and 6 *per Cent.* it will not be neceſſary to calculate thoſe Caſes, but ſuch only as require a Rate of Intereſt higher or lower, or intermediate; which will ſeldom happen, but in caſe it does, the Rule may eaſily be applied.

PROBLEM II.

The Values of two ſingle Lives being given, to find the Value of an Annuity granted for the Time of their joint continuance.

SOLUTION.

Let M be the Value of one Life, P the Value of the other, r the Rate of Intereſt; then the Value of an Annuity upon the two joint Lives will be $\frac{MP}{M+P-\overline{r-1}MP}$, in Words thus;

Multiply together the Values of the two Lives, and reſerve the Product.

Let that Product be again multiplied by the Intereſt of 1 l. *and let that new Product be ſubtracted from the Sum of the Values of the Lives, and reſerve the Remainder.*

Divide the firſt Quantity reſerved by the ſecond, and the Quotient will expreſs the Value of the two joint Lives.

Thus, ſuppoſing one Life of 40 Years of Age, the other of 50, and Intereſt at 5 *per Cent.* The Value of the firſt Life will be found in the Tables to be 11.83, the Value of the ſecond 10.35, the Product will be 122.4405, which Product muſt be reſerved.

Multiply this again by the Intereſt of 1 *l. viz.* by 0.05, and this new Product will be 6.122025.

This new Product being ſubtracted from the Sum of the Lives which is 22.18, the Remainder will be 16.057975, and this is the ſecond Quantity reſerved.

Now dividing the firſt Quantity reſerved by the ſecond, the Quotient will be 7.62 nearly; and this expreſſes the Values of the two joint Lives.

If the Lives are equal, the Canon for the Value of the joint Lives will be ſhortened and be reduced to $\frac{M}{2-\overline{r-1}\times M}$, which in words may be thus expreſſed;

Take the Value of one Life, and reſerve that Value.

Multiply this Value by the Intereſt of 1 l. *and then ſubtract the Product from the Number* 2, *and reſerve the Remainder.*

Divide the firſt Quantity reſerved by the ſecond, and the Quotient will expreſs the Value of the two equal joint Lives.

Thus, ſuppoſing each Life to be 45 Years of Age, and Intereſt at 5 *per Cent.*

The Value of one Life will be found to be 11.14, the firſt Quantity reſerved.

This being multiplied by 0.05 the Intereſt of 1 *l.* the Product will be 0.557.

This Product being ſubtracted from the Number 2, the Remainder will be 1.443, the ſecond Quantity reſerved.

Divide the firſt Quantity reſerved *viz.* 11.14; by the ſecond, *viz.* 1.443, and the Quotient 7.72 will be the Value of the two joint Lives, each of 45 Years of Age.

PROBLEM III.

The Values of three ſingle Lives being given, to find the Value of an Annuity for the Time of their joint continuance.

SOLUTION.

Let M, P, Q, be the reſpective Values of the ſingle Lives, then the Value of the three joint Lives will be $\frac{MPQ}{MP+MQ+PQ-2dMPQ}$, ſuppoſing d to repreſent the Intereſt of 1 *l.* in words thus;

Multiply the Values of the ſingle Lives together, and reſerve the Product.

Let that Product be multiplied again by the Intereſt of 1 l. *and let the Double of that new Product be ſubtracted from the Sum of the ſeveral Products of the Lives taken two and two, and reſerve the Remainder.*

Divide the firſt Quantity reſerved by the ſecond, and the Quotient will be the Value of the three joint Lives.

Thus, ſuppoſing one Life to be worth 13 Years Purchaſe, the ſecond 14, the third 15, and Intereſt at 4 *per Cent.* the Product of the three Lives will be 2730, which being multiplied by the Intereſt of 1 *l. viz.* by 0.04, the new Product will be 109.20, whereof the double is 218.40: Now the Product of the firſt Life by the ſecond is 182; the Product of the firſt Life by the third is 195; and the Product of the ſecond Life by the third is 210, the Sum of all which is 587; from which ſubtracting the Number 218.40 found above, the Remainder will be 368.60, by which the Product of the three Lives, *viz.* 2730 being divided, the Quotient 7.41 will be the Value of the three joint Lives.

But if the three Lives were equal, the general Expreſſion of the Value of the joint Lives will be much ſhorter: for let M repreſent the Value of one Life, d the Intereſt of 1 *l.* then the Value of the three joint Lives will be $\frac{M}{3-2dM}$, in Words thus;

Take the Value of one Life, and reſerve it, multiply this Value by the Intereſt of 1 l. *and double the Product.*

Subtract this double Product from the number 3, *and reſerve the Remainder.*

Divide the firſt Quantity reſerved by the ſecond, and the Quotient will be the Value of the three joint Lives.

Thus, ſuppoſing three equal Lives each worth 14 Years Purchaſe, reſerve the Number 14.

Multiply this by 0.04, Intereſt of 1 *l.* the Product will be 0.56, which being doubled, will be 1.12.

This being ſubtracted from the Number 3, the Remainder will be 1.88, which is the ſecond Quantity to be reſerved.

Divide 14, the firſt Quantity reſerved by the ſecond 1.88, and the Quotient 7.44 will be the Value of the three joint Lives.

From the two laſt Examples it appears, that in eſtimating the Values of joint Lives, it would be an Error to ſuppoſe that they might be reduced to an Equality, by taking a Mean Life betwixt the longeſt and ſhorteſt, for altho' 14 is a Medium betwixt 13 and 15, yet an Annuity upon thoſe three joint Lives was found to be 7.41, whereas ſuppoſing them to be each 14 Years Purchaſe, the Value is 7.44; it is true that the Difference is ſo ſmall, that it might be neglected, yet this ariſes meerly from a near Equality in the Lives; for if there had been a greater Inequality, the Concluſion would have conſiderably varied.

Before I come to the fourth Problem, I think it proper to explain the Meaning of ſome Notations which I make uſe of, in order to be as clear and conciſe as I can.

I denote the Value of an Annuity upon two joint Lives, whoſe ſingle Values are M and P by $\overline{MP}$, which ought carefully to be diſtinguiſhed from the Notation MP; this laſt denoting barely the Product of one Value multiplied by the other, whereas $\overline{MP}$ ſtands for what was denoted in our ſecond Problem by $\frac{MP}{M+P-\overline{r-1}\,MP}$.

In the ſame manner, the Value of an Annuity upon the three joint Lives whoſe ſingle Values are M, P, Q, is denoted by $\overline{MPQ}$, which is equivalent to what has been expreſſed in the third Problem by $\frac{MPQ}{MP+MQ+PQ-2d\,MPQ}$.

This being premiſed, I proceed to the fourth Problem.

PROBLEM IV.

The Values of two ſingle Lives being given, to find the Value of an Annuity upon the longeſt of them, that is, to continue ſo long as either of them is in being.

SOLUTION.

Let M be the Value of one Life, P the Value of the other, $\overline{MP}$ the Value of the two joint Lives, then the Value of the longeſt of the two Lives will be $M+P-\overline{MP}$. In Words thus;

From the Sum of the Values of the ſingle Lives, ſubtract the Value of the joint Lives, and the Remainder will be the Value of the longeſt.

Let us ſuppoſe two Lives, one worth 13 Years Purchaſe, the other 14, and Intereſt at 4 *per Cent.* The Sum of the Values of the Lives is 27, the Value of the two joint Lives by the Rules before given, will be found 9.23. Now, ſubtracting 9.23 from 27, the Remainder 17.77 is the Value of the longeſt of the two Lives.

If the two Lives are equal, the Operation will be ſomething ſhorter.

But it is proper to obſerve in this place, that if ſeveral equal Lives are concerned in an Annuity, I commonly denote one ſingle Life by M', two joint Lives by M'', three joint Lives by M''', and ſo on; ſo that the Rule for an Annuity to be granted till ſuch Time as either of the equal Lives is in being may be expreſſed by $2M'-M''$.

PROBLEM V.

The Values of three ſingle Lives being given, to find the Value of an Annuity upon the longeſt of them.

SOLUTION.

Let M, P, Q, be the Values of the ſingle Lives, $\overline{MP}, \overline{MQ}, \overline{PQ}$, the Values of all the joint Lives combined two and two, $\overline{MPQ}$ the Value of three joint Lives, then the Value of an Annuity upon the longeſt of them is $M+P+Q-\overline{MP}-\overline{MQ}-\overline{PQ}+\overline{MPQ}$, in Words thus;

Take the Sum of the three ſingle Lives, from which Sum ſubtract the Sum of all the joint Lives combined two and two, then to the Remainder add the Value of the three joint Lives, and the Reſult will be the Value of the longeſt of the three Lives.

Thus, Suppoſing the ſingle Lives to be 13, 14, and 15 Years Purchaſe, the Sum of the Values will be 42; the Values of the firſt and ſecond joint Lives is 9.24, of the firſt and third 9.05, of the ſecond and third 10.18, the Sum of all which is 29.06 which being ſubtracted from the Sum of the Lives found before, *viz.* 42, the Remainder will be 12.94, to which adding the Value of the three joint Lives 7.41, the Sum 20.35 will be the Value of the longeſt of the three joint Lives.

But if the three Lives are equal, the Rule for the Value of the Life that remains laſt is $3M' - 3M'' + M'''$.

Of REVERSIONS.

PROBLEM VI.

Suppoſe A *is in Poſſeſſion of an Annuity, and that* B *after the Deceaſe of* A *is to have the Annuity for him, and his Heirs for ever, to find the preſent Value of the Reverſion.*

SOLUTION.

Let M be the Value of the Life in Poſſeſſion, r the Rate of Intereſt, then the preſent Value will be $\frac{1}{r-1} - M$, that is, *from the Value of the Perpetuity, ſubtract the Value of the Life in Poſſeſſion, and the Remainder will be the Value of the Reverſion.*

Thus, Suppoſing that A is 50 Years of Age, an Annuity upon his Life, Intereſt at 5 *per Cent.* would be 8.39, which being ſubtracted from the Perpetuity 20, the Remainder will be 11.61, which is the preſent Value of the Expectation of B.

In the ſame manner, ſuppoſing that C were to have an Annuity for him and his Heirs for ever, after the Lives of A and B, then from the Perpetuity ſubtracting the Value of the longeſt of the two Lives of A and B, the Remainder will expreſs the Value of C's Expectation.

Thus, Suppoſing the Ages of A and B be 40 and 50, the Value of an Annuity upon the longeſt of theſe two Lives would be found by the 4th Problem to be 14.56; and this being ſubtracted from the Perpetuity 20, the Remainder is 5.44, which is the Value of C's Expectation, and the Rule will be the ſame in any other Caſe that may be propoſed.

PROBLEM VII.

Suppoſing that A *is in Poſſeſſion of an Annuity for his Life, and that* B *after the Life of* A, *ſhould have an Annuity for his Life only; to find the Value of the Life of* B *after the Life of* A.

This Caſe ought carefully to be diſtinguiſhed from the Caſe of the 6th Problem; for in that Problem, altho' the Expectant *B* ſhould die before *A*, ſtill the Heirs of *B* have the Reverſion; but in the Caſe of the preſent Problem, if *B* dies before *A*, the Heirs of *B* have no Expectation.

SOLUTION.

Let *M* be the Value of the Life of the preſent Poſſeſſor, *P* the Value of the Life of the Expectant, then the Value of his Expectation is $P - \overline{MP}$. In Words thus;

From the preſent Value of the Life of B, *ſubtract the preſent Value of the joint Lives of* B *and* A, *and the Remainder will be the Value of* B's *Expectation.*

The Reaſon of which Operation is very plain, for if *B* were now to begin to receive the Annuity, it would be worth to him the Sum *P* in preſent Value; but as he is to receive nothing during the joint Lives of himſelf and *A*, the preſent Value of their two joint Lives ought to be ſubtracted from the Value of his own Life.

PROBLEM VIII.

To find the Value of one Life after two.

Thus, Suppoſe A *in Poſſeſſion of an Annuity for his Life, that* B *is to have his Life in it after* A, *and that* C *is likewiſe to have his Life in it after* B, *but ſo that* B *dying before* A, C *ſucceeds* A *immediately; to find the Value of* C's *Expectation.*

SOLUTION.

Let *M*, *P*, *Q*, be the reſpective Values of the Lives of *A*, *B*, *C*, then the Value of *C*'s Expectation is $\begin{array}{l} Q - \overline{MQ} + \overline{MPQ} \\ \quad\;\; -\overline{PQ} \end{array}$, which in Words is thus expreſſed;

From the preſent Value of the Life of C, *ſubtract the Sum of the joint Lives of himſelf and* A, *and of himſelf and* B, *and to the Remainder add the Sum of the three joint Lives, and the Reſult of theſe Operations will expreſs the preſent Value of the Expectation of* C.

PROBLEM IX.

If A, B, C *agree among themſelves to buy an Annuity to be by them equally divided, whilſt they live together, then after the Deceaſe of one of them, to be equally divided between the two Survivors, then to belong entirely to the laſt Survivor for his Life; to find what each of them ought to contribute towards the Purchaſe.*

SOLUTION.

Let *M*, *P*, *Q*, be the reſpective Values of the Lives of *A*, *B*, *C*, then what *A* is to contribute, is

$$\begin{array}{l} M - \frac{1}{2}\overline{MP} + \frac{1}{3}\overline{MPQ} \\ \quad -\frac{1}{2}\overline{MQ} \end{array}$$

What *B* is to contribute, is

$$\begin{array}{l} P - \frac{1}{2}\overline{PM} + \frac{1}{3}\overline{MPQ} \\ \quad -\frac{1}{2}\overline{PQ} \end{array}$$

What *C* is to contribute, is

$$\begin{array}{l} Q - \frac{1}{2}\overline{QM} + \frac{1}{3}\overline{MPQ} \\ \quad -\frac{1}{2}\overline{QP} \end{array}$$

In Words thus;

From the Value of the Life of A, *ſubtract the half Sum of the Values of the joint Lives of himſelf and* B, *and of himſelf and* C, *and to the Remainder add* $\frac{1}{3}$ *of the Value of the three joint Lives, and the Sum will be what* A *is to contribute towards the Purchaſe.*

In like manner, from the Value of *B*'s Life ſubtract the half Sum of the Values of the joint Lives of himſelf and *A*, and of himſelf and *C*, and to the Remainder add $\frac{1}{3}$ of the Value of the three joint Lives, and the Sum will be what *B* is to contribute.

And again, from the Value of the Life of *C*, ſubtract the half Sum of the Values of himſelf and *A*, and of himſelf and *B*; then to the Remainder add $\frac{1}{3}$ of the Values of the three joint Lives, and this laſt Operation will ſhew what *G* is to contribute.

PROBLEM X.

Suppoſing three equal Lives of any Age given, for Inſtance 30, *and that upon the Failing of any one of them, that Life ſhall be immediately replaced, and I then receive a Sum* ſ *agreed upon, and that to Perpetuity for me and my Heirs; what is the preſent Value of that Expectation, and at what Intervals of Time, one with another, may I expect to receive the ſaid Sum?*

SOLUTION.

Imagine that there is an Annuity of 1 *l.* to be received as long as the three Lives are in being, and that the Preſent Value is M''', which Symbol we make uſe of to repreſent the Preſent Value of an Annuity upon three equal joint Lives; now, ſince each Life is ſuppoſed to be 30 Years of Age, and that the Rate of Intereſt is 5 *per Cent.* we ſhall find, by following the Directions given in Prob. III. that the Preſent Value of the three joint Lives is $7.64 = M'''$; this being fixed, the Preſent Value of all the Payments to be made to Eternity at equal Intervals of Time, will be $\frac{1-dM'''}{dM'''}$ × ſ, where the Quantity d ſignifies the Intereſt of 1 *l.* In words thus;

Multiply the Preſent Value of the three joint Lives, viz. 7.64, *by the Intereſt of* 1 l. *which in this Caſe is* 0.05, *and that Product, which is* 0.382, *muſt be reſerved.*

Subtract this Quantity from Unity, and the Remainder, viz. 0.618 *being divided by the Quantity reſerved, the Quotient will be* 1.62, *and this being multiplied by the Sum* ſ, *which we may ſuppoſe* 100 l. *the Product will be* 162 l. *and this is the preſent Value of all the Payments that will be made to Eternity, at equal Intervals of Time upon the failing of a Life, which is to be immediately replaced.*

As for the Intervals of Time after which thoſe Replacements will be made, they may be found thus;

Look in the ſeventh of our Tables for the Number 7.64, which is the Value of the three joint Lives, and over againſt it will be found the Number anſwering, which is between 9 and 10; and ſo it may be ſaid that the Replacements will be made at every Interval of about 9 or 10 Years.

But that Interval may be determined a little more accurately, by help of a Table of Logarithms, by taking the Logarithm of the Quantity $\frac{1}{1-dM'''}$ and dividing it by the Logarithm of r.

The Logarithm of $\frac{1}{1-dM'''}$ is 0.2090115; the Logarithm of r is 0.0211893; and the firſt being divided by the ſecond, the Quotient is 9.86, which ſhews that the Replacements will be made at Intervals a little more than $9\frac{3}{4}$ Years.

PROBLEM XI.

Suppoſing, as before, three equal Lives of 30, *and that the Lives are not to be renewed, till after the failing of any two of them, and that a Sum* p *is then to be received, and that perpetually, after the failing of two Lives, what is the preſent Value of that Expectation?*

SOLUTION.

Make $3M'' - 2M''' = A$, let the Intereſt of 1 *l.* be $= d$, then the preſent Value of that Expectation will be $\frac{1-dA}{dA} \times p$.

But to know the Intervals of Time after which the Lives will be filled up, take the Logarithm of the Quantity $\frac{1}{1-dA}$, and divide it by the Logarithm of r.

The Value of a ſingle Life of 30, Intereſt at 5 *per Cent.* is found in our Tables to be 13 Years Purchaſe, the Value M'' of two joint Lives by Problem II. is 9.63; and the Value M''' of three joint Lives by Problem III. is 7.64; then $3M'' - 2M'''$, or the Difference between the Triple of two joint Lives, and the Double of three joint Lives will be $13.59 = A$, then $\frac{1-dA}{dA} \times p$ will be found to be $0.473\,p$, and the Intervals of Time will be 23.32, that is, nearly $23\frac{1}{3}$ Years.

PROBLEM XII.

Suppoſing ſtill the Lives to be 30, *and that they are not to be renewed till after the Extinction of all three, and that a Sum* q *is then to be received, and that perpetually after every Renewal, what is the preſent Value of that Expectation?*

SOLUTION.

Make $3M' - 3M'' + M''' = B$, then the preſent Value of that Expectation will be $\frac{1-dB}{dB} \times q$; here B will be found to be 17.76, and conſequently $\frac{1-dB}{dB} \times q$ will be $= 0.121 \times q$.

And the Intervals of Time will be the Logarithm of the Quantity $\frac{1}{1-dB}$ divided by the Logarithm of r, which in this Caſe would be 44.87, that is, nearly 45 Years.

COROLLARY.

Hence it will be eaſy for the Proprietor of the Lives, to find which is moſt advantageous to him, to fill up a Life as ſoon as it is vacant, or not to fill up before the Vacancy of two, or to let them all drop before the Renewal.

REMARK.

It is not to be imagined that if Intereſt of Money was higher or lower than 5 *per Cent.* the Intervals of Time after which the Renewals are made, would be the ſame as they are now, for it will be found, that as Intereſt is higher, the Intervals will be ſhorter; and as it is lower, ſo the Intervals will be longer; yet one might make it an Objection to our Rules, that the length of Life would thereby ſeem to depend upon the Rate of Intereſt. To anſwer this Difficulty, it muſt be obſerved, that the calculating of Time imports no more, than that conſidering the Circumſtances of the Purchaſer and the Proprietor of the Lives, in reſpect to the Rate of Intereſt agreed upon, and the Sum to be given upon the Renewal of a Life, or Lives, the Proprietor makes the ſame Advantage of his Money, as if he had agreed with the Purchaſer, that he ſhould pay him a certain Sum of Money at equal Intervals of Time, for redeeming the

Riſque which he the Purchaſer runs of paying that Sum when the Life or Lives drop: but the real Intervals of Time will be ſhewn afterwards.

Altho' it ſeldom happens that in Contracts about Lives, any more than three are concerned, yet I hope it will not be diſpleaſing to our Readers to have this Speculation carried a little farther.

But as general Rules are beſt inculcated by particular Examples, I ſhall take the Caſe of five Lives, and expreſs the ſeveral Circumſtances of them in ſuch manner, as that they may be a ſure Guide in all other Caſes of the ſame kind, let the Number of Lives be what it will; let therefore the following Expreſſions be written,

$$\begin{array}{r} M''''' \\ 5M'''' - 4M''''' \\ 10M''' - 15M'''' + 6M''''' \\ 10M'' - 20M''' + 15M'''' - 4M''''' \\ 5M' - 10M'' + 10M''' - 5M'''' + 1M''''' \end{array}$$

The firſt Term M''''' repreſents properly the preſent Value of an Annuity upon five equal joint Lives, but from thence may be deduced the Time of their joint continuance, or the Time in which it may be expected that one of them will fail, it being as I have ſaid before, the Logarithm of $\frac{1}{1-dM'''''}$ divided by the Logarithm of r: however, for ſhortneſs ſake, I call for the preſent that Expreſſion the Time.

The two next Terms, $5M'''' - 4M'''''$, repreſent the Time in which two of the Lives will fail.

The three next Terms, $10M''' - 15M'''' + 6M'''''$, repreſent the Time in which three out of the five Lives will fail.

The four next, $10M'' - 20M''' + 15M'''' - 4M'''''$, repreſent the Time in which four out of the five Lives will fail.

The five next, $5M' - 10M'' + 10M''' - 5M'''' + 1M'''''$, repreſent the Time in which all the five Lives will be extinct.

Now the Law of the Generation of the Co-efficients is thus.

1°. Take all the Terms which are affected with the Mark M''''', beginning from the uppermoſt, with the Co-efficients $1 - 4 + 6 - 4 + 1$, which are the Terms of the Binomial $1 - 1$, raiſed to the fourth Power, which is leſs by one than the Number of Lives concerned.

2°. Take the Terms which are affected with the Mark M'''', and prefix to them in order, the product of the Number 5 by the Co-efficients $1 - 3 + 3 - 1$, which are the Terms of the Binomial $1 - 1$ raiſed to its Cube, that is, to a Power leſs by two than the Number of Lives concerned.

3°. Take all the Terms which are affected with the Mark M''', and prefix to them in order, the Product of the Number 10, multiplied by the Co-efficients $1-2+1$, which are the Terms of the Binomial $1-1$ raiſed to its Square, that is, to a Power leſs by three than the Number of Lives concerned.

4°. Take all the Terms which are affected with the Mark M'', and prefix to them the product of the Number 10, multiplied by the Terms of the Binomial $1-1$, raiſed to the Power whoſe Index is 1, that is to a Power leſs by four than the Number of Lives concerned.

5°. Take all the Terms which are affected with the Mark M', and prefix to them the Product of the Number 5, multiplied by the Binomial $1-1$, raiſed to a Power leſs by 5 than the Number of Lives concerned; which in this Caſe happening to be nothing, or 0, degegenerates barely into Unity.

As for the Multiplicators, conceiving that the Multiplicator of the firſt Term M'''' is 1, all the Multiplicators will be 1, 5, 10, 10, 5, which are all, except the laſt, the Coefficients of the Binomial $1+1$, raiſed to its fiſth Power, that is, to a Power equalling the Number of all the Lives.

N. B. The Exception here given, does not fall upon the Number 5, but upon the laſt Term of the fiſth Power, $1+5+10+10+5+1$, which laſt 1 is rejected.

Of SUCCESSIVE LIVES.

PROBLEM XIII.

If A *enjoys an Annuity for his Life, and at his Deceaſe has the Nomination of a Succeſſor* B, *who is alſo to enjoy the Annuity for his Life, to find the preſent Value of the two ſucceſſive Lives.*

SOLUTION.

Let the Values of the Lives be M and P; let d be the Intereſt of 1 *l.* then the Value of the two ſucceſſive Lives will be $M+P-dMP$.

But if the Succeſſor B was himſelf to have the Nomination of a Life Q; then the Value of the three ſucceſſive Lives would by $M+P+Q-d\times\overline{MP+MQ+PQ}+dd\times MPQ$.

But before I proceed, it is proper to obſerve that the Expreſſions *MP*, *MQ*, *PQ*, and *MPQ*, ſignify barely Products, which is conformable to the uſual Algebraic Notation; this I take notice of, for fear thoſe Expreſſions ſhould be confounded with others that I have made uſe of before, *viz.* $\overline{MP}$, $\overline{MQ}$, $\overline{PQ}$, and $\overline{MPQ}$, which denoted joint Lives.

But to compriſe under one general Rule all the poſſible Caſes that may happen about any Number of ſucceſſive Lives, it will be proper to expreſs it in Words at length, thus;

From the Sum of all the Lives, ſubtract the Sum of the Products of all the Lives combined two and two, which Sum of Products before they are ſubtracted, ought to be multiplied by the Intereſt of 1 l.

To this add the Sum of the Products of all the Lives taken three and three, but multiplied again by the Square of the Intereſt of 1 l.

From this ſubtract the Sum of the Products of all the Lives taken four and four, but multiplied again by the Cube of the Intereſt of 1 l. *and ſo on by alternate Additions and Subtractions ſtill obſerving that if there was occaſion to take the Lives five and five, ſix and ſix,* &c. *the Intereſt of* 1 l. *ought to be raiſed to the 4th Power, and to the 5th, and ſo on.*

But all thoſe Operations would be very much contracted, if the Lives to be nominated were always of the ſame Age, for Inſtance 30: for ſuppoſe *M* to be the Value of an Annuity on an Age of 30, and *d* to be the Intereſt of 1 *l.* then the preſent Value of all the ſucceſſive Lives, of which the Number is *n*, would be $\frac{1-\overline{1-am}^{\,n}}{d}$. In Words thus;

Multiply the Value of one Life by the Intereſt of 1 l. *let the Product be ſubtracted from Unity, and let the Remainder be raiſed to that Power which anſwers to the Number of Lives; then this Power being again ſubtracted from Unity, let the Remainder be divided by the Intereſt of* 1 l. *and the Quotient will be the preſent Value of all the ſucceſſive equal Lives.*

And again, if the Number of thoſe Lives were infinite, the Sum would barely be $\frac{1}{d}$.

PROBLEM XIV.

Of a Perpetual Advowſon.

1°. I ſuppoſe that at the Time of the Demiſe of the Incumbent, the Patron would receive the Sum *ſ*, for alienating his Right of the next Preſentation, if the Law did not forbid the Alienation in that Circumſtance of Time.

2°. I ſuppoſe that when this Right is transferred, the Age of the Incumbent is ſuch, that an Annuity upon his Life would be worth M Years Purchaſe, when the Intereſt of 1 *l.* is d.

This being ſuppoſed, the Right of the next Preſentation is worth $\overline{1-dM}\times ſ$, and the Right of Patronage, or perpetual Recurrency of the like Circumſtances to Eternity, would be worth $\frac{1-dM}{dM}\times ſ$. In words thus;

Take the preſent Value of the Life of the Incumbent, and multiply it by the Intereſt of 1 l. *and reſerve the Product.*

Subtract this Product from Unity, and let the Remainder be multiplied by the Sum expected ſ, *and the new Product will ſhew the Right of the next Preſentation; let alſo this be reſerved.*

Then divide the ſecond Quantity reſerved by the firſt, and the Quotient will ſhew the preſent Value of the Right of Patronage, or perpetual Recurrency.

Thus, ſuppoſing the Life of the Incumbent worth 8 Years Purchaſe, the Rate of Intereſt 5 *per Cent.* and the Sum $ſ$ to be 100 *l.* the Right of the next Preſentation would be worth 60 *l.* and the Right of perpetual Recurrency 150 *l.*

PROBLEM XV.

Of a Copy-hold.

Suppoſing that every Copy-hold Tenant pays to the Lord of the Manor a certain Fine on Admittance, and that every Succeſſor does the like; to find the Value of the Copy-hold computed from the Time of a Fine being paid, independently from the Fine that may be given on Alienation.

SOLUTION.

I ſuppoſe that the Value of the Life of the preſent Tenant, and the Life of every future Succeſſor when he comes to Poſſeſſion is the ſame; this being admitted, let M be the Value of a Life, d the Intereſt of 1 *l.* and $ſ$ the Fine to be paid, then the preſent Value of the

Copy-hold will be $\frac{1-dM}{dM}\times ſ$: and this Expreſſion being exactly the ſame as that whereby the Right of Patronage has been determined, needs no Explanation in Words.

Only it is neceſſary to obſerve, that the Sum $ſ$ paid in Hand being added to this, will make the Canon ſhorter, and will be reduced to $\frac{ſ}{dM}$, which may be expreſſed thus in Words.

Divide the Fine by the Product of the Life, multiplied by the Intereſt of 1 l.

Thus, if the Life of a Tenant is worth 12 Years Purchaſe, and the Fine to be paid on Admittance 56 *l.* and alſo the Rate of Intereſt 5 *per Cent.* then the preſent Value of the Copy-hold is 93 $\frac{1}{3}$ *l.*

PROBLEM XVI.

A *borrows a certain Sum of Money, and gives Security that it ſhall be repaid at his Deceaſe with the Intereſts; to fix the Sum which is then to be paid.*

SOLUTION.

Let the Sum borrowed be $ſ$, the Life of the Borrower M Years purchaſe, d the Intereſt of 1 *l.* then the Sum to be paid at *A*'s Deceaſe will be $\frac{ſ}{1-dM}$; thus, ſuppoſing $ſ = 800$, $M = 11.83$, $d = 0.05$, then $\frac{ſ}{1-dM}$ would be found $= 1958$ *l:* in the ſame manner, if the Sum to be paid at *A*'s Deceaſe, was to be an Equivalent for his Life, unpaid at the Time of the Purchaſe, that Sum would be $\frac{M}{1-dM} = 2895$ *l.* Supppoſing the Annuity received to be 100 *l.* as alſo the Life of *A* 11.83 Years Purchaſe.

PROBLEM XVII.

A *borrows a Sum* ſ, *payable at his Deceaſe, but with this Condition, that if he dies before* B, *then the whole Sum is to be loſt to the Lender; to find what* A *ought to pay at his Deceaſe in caſe he ſurvives* B.

SOLUTION.

Let us ſuppoſe, as before, that A is 40 Years of Age, that the Sum borrowed is 800 *l.* and that Intereſt of Money is 5 *per Cent.* Farther, let it be ſuppoſed that B is 70 Years of Age, then, 1°. determine what A ſhould pay at his Deceaſe, if the Life of B was not concerned; by the Solution of the preceding Problem, we find the Sum to be 1958 *l.* But we ought to conſider that the Lender having a Chance to loſe his Money, there ought to be a Compenſation for the Riſque he runs, which is founded on the poſſibility of a Man of ſeventy outliving a Man of forty. Now, by the Rules to be delivered in the next Problem, we ſhall find that the Probability of that Contingency is meaſured by the Fraction $\frac{4}{23}$, and therefore the Probability of the youngeſt Life's ſurviving the oldeſt is $\frac{19}{23}$. Now this being the Meaſure of the Probability which the Lender has of being repaid, the Sum 1958 ought to be increaſed in the proportion of 23 to 19, which will make it to be 2370 *l.* nearly.

Of the Probabilities of Survivorſhip.

PROBLEM XVIII.

Any Number of Lives being given, to find their Probability of Survivorſhip.

SOLUTION.

Let A, B, C, D, *&c.* be the Lives, whereof A is ſuppoſed to be the youngeſt, B the next to it, C the next, *&c.* and ſo the laſt the oldeſt.

Let n, p, q, s, t, *&c.* be the reſpective Intervals intercepted between the Ages of thoſe Lives, and the Extremity of old Age ſuppoſed at 86; then the Probabilities of any one of thoſe Lives ſurviving all the reſt, will be

$$\begin{array}{r|cccccc}
\text{for } A & 1 & -\frac{p}{2n} & -\frac{qq}{6np} & -\frac{s^3}{12npq} & -\frac{t^4}{20npqs} \\
B & & +\frac{p}{2n} & -\frac{qq}{6np} & -\frac{s^3}{12npq} & -\frac{t^4}{20npqs} \\
C & & & +\frac{qq}{3np} & -\frac{s^3}{12npq} & -\frac{t^4}{20npqs} \\
D & & & & +\frac{s^3}{4npq} & -\frac{t^4}{20npqs} \\
E & & & & & +\frac{t^4}{5npqs} \\
\&c. & & & & &
\end{array}$$

Here ſome few things may be obſerved.

1°. That the Probability of the youngeſt Life ſurviving all the reſt, always begins with Unity, and that it is expreſſed by ſo many Terms as there are Lives concerned.

2°. That the Probabilities of the other Lives ſurviving all the reſt, are always expreſſed each by one Term leſs than the preceding.

3°. That each firſt Term of thoſe whereby each Probability is expreſſed, is always the Sum of all the other Terms ſtanding above it.

4°. That the Numbers 2, 6, 12, 20, 30, *&c.* made uſe of in the Denominators of the Fractions are generated by the Multiplication of the following Numbers, 1 × 2, 2 × 3, 3 × 4, 4 × 5, *&c.* It would take up too much room to explain this general Rule in Words at length, for which Reaſon I ſhall content my ſelf with explaining only the Caſes of two and three Lives, which are the moſt neceſſary.

And, Firſt, *if there be two Lives of a given Age, ſuch as* 40 *and* 70, *take their Complements of Life, which as I have explained before, are the Differences between* 86 *and the reſpective Ages, thoſe Complements therefore are* 46 *and* 16.

Divide the ſhorteſt Complement by the Double of the Longeſt, and the Quotient will expreſs the Probability of the oldeſt Life ſurviving the youngeſt.

Thus in the preſent Caſe, the ſhorteſt Complement being 16, and the double of the longeſt being 92, I divide 16 by 92, and the Quotient $\frac{16}{92}$ or $\frac{4}{23}$ will expreſs the Probability required.

Subtract this Fraction from Unity, and the Remainder $\frac{19}{23}$ will expreſs the Probability of the youngeſt Life ſurviving the oldeſt.

So that the Odds of the youngeſt Life ſurviving the oldeſt, are 19 to 4.

The Caſe of three Lives is thus: Suppoſe there are three Lives of a given Age, ſuch as 40, 45, and 60; take their reſpective Complements of Life, which are 46, 41, 26, then divide the Square of the ſhorteſt Complement by 3 times the Product of the other two, and the Quotient will expreſs the Probability of the oldeſt Life ſurviving the other two.

Divide the middlemoſt Complement by the Double of the greateſt, and from the Quotient ſubtract the Square of the leaſt divided by 6 Times the Product of the other two, and the Remainder will expreſs the Probability of the middlemoſt Life ſurviving the other two.

Subtract the Sum of the two foregoing Probabilities from Unity, and the Remander will expreſs the Probability of the youngeſt Life ſurviving the other two.

Thus in the Caſe propoſed, the Probability of the oldeſt Life ſurviving the other two, will be found $\frac{676}{5658} = \frac{3}{25}$ nearly.

The Probability of the middlemoſt Life ſurviving the other two will be $\frac{4367}{11316} = \frac{5}{13}$ nearly.

The Probability of the youngeſt Life ſurviving the other two will be $\frac{3}{5}$ nearly.

PROBLEM XIX.

Any Number of Lives being given, to find the Probability of the Order of their Survivorſhip.

SOLUTION.

Suppoſe the three Lives to be thoſe of *A*, *B*, *C*, and that it be required to aſſign the Probability of Survivorſhip as limited to the Order in which they are written, ſo that *A* ſhall both ſurvive *B* and *C*, and *B* alſo ſurvive *C*. This being ſuppoſed, let n, p, q, repreſent the reſpective Complements of Life, of the youngeſt, middlemoſt, and oldeſt, then the Probabilities of the ſix different Orders that there are in three things, will be as follows;

A, *B*, *C*	$1 - \frac{p}{2n} - \frac{q}{2p} + \frac{qq}{6np}$
A, *C*, *B*	$\frac{q}{2p} - \frac{qq}{3np}$
B, *A*, *C*	$\frac{p}{2n} - \frac{q}{2n} + \frac{qq}{6np}$
B, *C*, *A*	$\frac{q}{2n} - \frac{qq}{3np}$
C, *A*, *B*	$\frac{qq}{6np}$
C, *B*, *A*	$\frac{qq}{6np}$

In Words thus;

1°. *Divide the middlemoſt Complement by the double of the greateſt, and let the Quotient be ſubtracted from Unity.*

2°. *From that Remainder ſubtract again the Quotient of the ſhorteſt Complement divided by the Double of the Middlemoſt.*

3°. *To that new Remainder add the Quotient ariſing from the Square of the ſhorteſt Complement divided by ſix times the Product of the greateſt and middlemoſt multiplied together, and this laſt Sum will expreſs the Probability of the firſt Order.*

The probability of the Second will be found thus;

1°. *Divide the ſhorteſt Complement by the double of the middlemoſt, and reſerve the Quotient.*

2°. *Divide the Square of the ſhorteſt by three times the Product of the longeſt Complement, multiplied by the Middlemoſt, and reſerve the new Quotient.*

3°. *Let the ſecond Quotient be ſubtracted from the firſt, and the Remainder will expreſs the Probability of the happening of the ſecond Order.*

The Probability of the third Order will be found as follws.

1°. *Divide the middlemoſt Complement by the Double of the Greateſt, and reſerve the Quotient.*

2°. *Divide the ſhorteſt Complement by the Double of the longeſt, and reſerve the Quotient.*

3°. *Divide the Square of the ſhorteſt Complement by ſix times the Product of the longeſt and middlemoſt multiplied together, and reſerve the Quotient.*

4°. *From the firſt Quotient reſerved, ſubtract the ſecond; then to the Remainder add the Third, and the Reſult of theſe Operations will expreſs the Probability of the third Order.*

The Probability of the fourth Order will be found thus,

1°. *Divide the ſhorteſt Complement by the Double of the longeſt, and reſerve the Quotient.*

2°. *Divide the Square of the ſhorteſt Complement by three Times the Product of the longeſt and middlemoſt, and reſerve the new Quotient.*

3°. *From the firſt Quotient reſerved, ſubtract the ſecond, and the Remainder will expreſs the Probability of the fourth Order.*

The fifth Order will be found as follows.

Divide the Square of the ſhorteſt Complement by ſix times the Product of the longeſt and middlemoſt, multiplied together, and the Quotient will expreſs the Probability required.

The Probability of the laſt Order is the ſame as that of the fifth.

PROBLEM XX.

D, *whilſt in Health, makes a Will, whereby he bequeaths* 500 l. *to* E, *and* 300 l. *to* F. *with this Condition, that if either of them dies before him, the whole is to go to the Survivor of the two; what are the Values of the Expectations of* E *and* F, *eſtimated from the time that the Will was writ?*

SOLUTION.

Suppoſe *D* to be 70 Years of Age, *E* 36, and *F* 45; ſuppoſe alſo that *d* repreſents the Intereſt of 1 *l.* when Intereſt is at 5 *per Cent.*

An Annuity upon the Life of *D* is worth 5.77, as appears from our Tables, which Value we may call *M.*

Wherefore if it was ſure that *D* would die before either of them, the Expectation of *E* upon that Account, would be worth in preſent Value $\overline{1-dM}\times 500$, and the Expectation of *F*, $\overline{1-dM}\times 300$; which being reduced to Numbers, are reſpectively 355 *l.* 15 *s.* and 213 *l.* 9 *s.*

But as this depends on the Probability of *D*'s dying firſt, we are to look for that Probability, which is compoſed of two Parts, that is, when the Order of Survivorſhip is either *E, F, D,* or *F, E, D*; now the Order *E, F, D,* is the ſame as *A, B, C,* in the preceding Problem, whereof the Probability is $1-\frac{p}{2n}-\frac{q}{2p}+\frac{qq}{6np}$, and the Order *F, E, D,* is the ſame as *B, A, C,* whereof the Probability is $\frac{p}{2n}-\frac{q}{2n}+\frac{qq}{6np}$, and the Sum of thoſe Probabilities, *viz.* $1-\frac{q}{2p}-\frac{q}{2n}+\frac{qq}{3np}$, will expreſs the Probability of *D*'s dying before them both.

Now the Ages being given, their Complements of Life will alſo be given, ſo that n will be found $=50$, $p=41$, $q=16$; for which reaſon the Probability juſt now ſet down being expreſſed in Numbers, will be 0.6865, and this being multiplied by the Expectations before found, *viz.* 315 *l.* 15 *s.* and 213 *l.* 9 *s.* will produce 244 *l.* 3 *s.* 5 *d.* and 146 *l.* 10 *s.* 8 *d* and theſe Sums expreſs the preſent Expectations of *E* and *F*, ariſing from the Proſpect of *D*'s dying before either of them.

But both *E* and *F* have a farther Expectation; which, in reſpect to *E*, is, that he ſhall ſurvive *D*, and that *D* ſhall ſurvive *F*, in which Caſe he obtains 800 *l.* but this not being to be obtained before the Deceaſe of *D*, is reduced in preſent Value to 569 *l.* 4 *s.* Now the Probability of obtaining this anſwers to the Order, *A*, *C*, *B*, in the preceding Problem, which is expreſſed by $\frac{q}{2p} - \frac{qq}{3np} = 0.1535$; and therefore multiplying the Sum 569 *l.* 4 *s.* by 0.1535, the Product will be 87 *l.* 7 *s.* 5 *d.* and this will be the ſecond Part of *E*'s Expectation, which being joined with the firſt Part found before, *viz.* 244 *l.* 3 *s.* 5 *d.* the Sum will be 331 *l.* 10 *s.* 10 *d.* which is the total Expectation of *E*, or the preſent Sum he might juſtly expect, if he would ſell his Right to another.

In the ſame manner the total Expectation of *F* will be found to be 213 *l.* 18 *s.* 6 *d.*

Otherwiſe, and more exactly, thus;

1. Let the Value of an Annuity of 40 *l.* for *D*'s Life, be taken off; which reduces the Sum to *l.* 569.2. as above.

2. The Heirs of *D* have likewiſe a demand upon this laſt Sum, for the Contingency of his outliving both the Legatees; which is implied tho' not expreſſed in the Queſtion. Subtract therefore from the Value of the longeſt of the 3 Lives *D*, *E*, *F*, which, by Prob. V, is 15.477, the Value of the longeſt of the two Lives *E*, *F*; which, by Prob. IV, is 15.197; and the Remainder 0.28, *D*'s Survivorſhip due to the Heirs, taken from *l.* 569.2, conſidered as 20 Years Purchaſe, or the Perpetuity, reduces it to *l.* 561.23.

3. This Sum, now clear of all demands, might be paid down immediately to *E* and *F*, in the proportions of 5 and 3, according to the Will; were their Ages equal. And altho' they are not, we ſhall ſuppoſe that *D*, or his Executor named, pays it them in that manner; the Share of *E* being *l.* 350.77, and that of *F*, *l.* 210.46. leaving them to adjuſt their Pretenſions, on account of Age, between themſelves.

4. In order to which; the Sums which *E* and *F* have received being called *G* and *L*, reſpectively; let the Value of *E*'s Survivorſhip after *D* and *F*, found as above, be denoted by *e*, and that of *F* after *D* and *E* by *f*: Then thoſe Values, *e* and *f*, will repreſent the Chances, or Claims, which *E* and *F* have upon each other's Sums *L* and *G*. And therefore the Ballance of their Claims is $\frac{eL - fG}{e+f}$; due by *F* or *E* as the Sign is poſitive or negative.

As in our Example, e and f being 3.26, 2.269 reſpectively, E muſt refund to F $\left(\frac{eL-fG}{e+f}=-\right)$ *l.* 19.857; and the juſt Values of their Legacies will be *l.* 330. 18 *s.* and *l.* 230. 6 *s.*

This laſt Computation is to be uſed when the Teſtator D is not very old, or the Ages of E and F are conſiderably different; or when both theſe Conditions obtain: For in thoſe Caſes, the Ratio of the *Probabilities* of Survivorſhip will differ ſenſibly from that of the *Values of the Probabilities* reckoned in Years purchaſe. And the like caution is to be obſerved in all ſimilar Caſes.

Of the Expectations of Life.

I call that the Expectation of Life, the Time which a Perſon of a given Age may juſtly expect to continue in being.

I have found by a Calculation deduced from the Method of Fluxions, that upon Suppoſition of an equable Decrement of Life, the Expectation of Life would be expreſſed by $\frac{1}{2}n$, ſuppoſing n to denote its Complement.

However, if that Interval be once attained, there ariſes a new Expectation of $\frac{1}{4}n$, and afterwards of $\frac{1}{8}n$, and ſo on. This being laid down, I ſhall proceed farther.

PROBLEM XXI.

To find the Expectation of two joint Lives, that is, the Time which two Lives may expect to continue together in being.

SOLUTION.

Let the Complements of the Lives be n and p, whereof n be the longeſt and p the ſhorteſt, then the Expectation of the two joint Lives, will be $\frac{1}{2}p-\frac{pp}{6n}$, in Words thus.

From $\frac{1}{2}$ *the ſhorteſt Complement, ſubtract the 6th Part of its Square, divided by the greateſt, the Remainder will expreſs the Number of Years ſought.*

Thus, ſuppoſing a Life of 40, and another of 50, the ſhorteſt Complement will be 36, the greateſt 46, $\frac{1}{2}$ of the ſhorteſt will be

18,

18, the Square of 36 is 1296, whereof the ſixth Part is 216, which being divided by 46, the Quotient will be $\frac{216}{46} = 4.69$ nearly; and this being ſubtracted from 18, the Remainder 13.31 will expreſs the Number of Years due to the two joint Lives.

COROLLARY.

If the two Lives be equal, the Expectation of the two joint Lives will be $\frac{1}{3}$ part of their common Complement.

PROBLEM XXII.

Any Number of Lives being given, whether equal or unequal, to find how many Years they may be expected to continue together.

SOLUTION.

1°. Take $\frac{1}{2}$ of the ſhorteſt Complement.

2°. Take $\frac{1}{6}$ part of the Square of the ſhorteſt, which divide ſucceſſively by all the other Complements, then add all the Quotients together.

3°. Take $\frac{1}{12}$ part of the Cube of the ſhorteſt Complement, which divide ſucceſſively by the Product of all the other Complements, taken two and two.

4°. Then take $\frac{1}{20}$ part of the Biquadrate of the ſhorteſt Complement, which divide ſucceſſively by the Products of all the other Complements, taken three and three, and ſo on.

5°. Then from the Reſult of the firſt Operation, ſubtract the Reſult of the ſecond, to the Remainder add the Reſult of the third, from the Sum ſubtract the Reſult of the fourth, and ſo on.

6°. The laſt Quantity remaining after theſe alternate Subtractions and Additions, will be the thing required.

N. B. The Diviſors 2, 6, 12, 20, *&c.* are the Products of 1 by 2, of 2 by 3, of 3 by 4, of 4 by 5, *&c.*

COROLLARY.

If all the Lives be equal, add Unity to the Number of Lives, and divide their common Complement by that Number thus increaſed by

Unity, and the Quotient will always expreſs the Time due to their joint Continuance.

PROBLEM XXIII.

Two Lives being given, to find the Number of Years due to the Longeſt.

SOLUTION.

From the Sum of the Years due to each Life, ſubtract the Number of Years due to their Joint Continuance, the Remainder will be the Number of Years due to the Longeſt, or Survivor of them both.

Thus, ſuppoſing a Life of 40, and another of 50, the Number of Years due to the Life of 40, is 23; the Number of Years due to the Life of 50, is 18; from the Sum of 23 and 18, *viz.* 41, ſubtract 13.31 due to their joint Continuance, the Remainder 27.69 will be the Time due to the longeſt.

COROLLARY.

If the Lives be equal, then $\frac{2}{3}$ of their common Complement will be the Number of Years due to the Survivor.

Thus, ſuppoſing two Lives of 50, then their Complement will be 36; whereof two thirds will be 24; which is the Time due to the Survivor of the two.

PROBLEM XXIV.

Any Number of Lives being given, to find the Number of Years due to the Longeſt.

SOLUTION.

Let the Years due to each Life be reſpectively denoted by M, P, Q, S, *&c.* then let the joint Lives, taken two and two, be denoted by $\overline{MP}$, $\overline{MQ}$, $\overline{MS}$, $\overline{PQ}$, &c. let alſo the joint Lives, taken three and three be denoted by $\overline{MPQ}$, $\overline{MPS}$, $\overline{MQS}$, $\overline{PQS}$, &c. Moreover, let the joint Lives, taken four and four, be denoted by $\overline{MPQS}$, &c. then if there be three Lives, the Time due to the longeſt will be

$$\begin{array}{l} M-\overline{MP}+\overline{MPQ} \\ +P-\overline{MQ} \\ +Q-\overline{PQ} \end{array}$$

But if all the Lives be equal, let n be their common Complement, then the Time due to the longeſt, will be $\frac{3}{4}n$.

If there be four Lives, the Time due to the longeſt will be

$$\begin{array}{llll} M & -\overline{MP} & +\overline{MPQ} & -\overline{MPQS} \\ +P & -\overline{MQ} & +\overline{MPS} & \\ +Q & -\overline{MS} & +\overline{MQS} & \\ +S & -\overline{PQ} & +\overline{PQS} & \\ & -\overline{PS} & & \\ & -\overline{QS} & & \end{array}$$

But if all the Lives be equal, the Time due to the longeſt will be expreſſed by $\frac{4}{5}$ of their common Complement.

Univerſally, if the common Complement of equal Lives be n, and the Number of Lives p, the Number of Years due to the Longeſt of them will be $\frac{p}{p+1} \times n$.

PROBLEM XXV.

Any Number of equal Lives being given, to find the Time in which one, or two, or three, &c. *of them will fail.*

SOLUTION.

Let n be their common Complement, p the Number of all the Lives, q the Number of thoſe which are to fail, then $\frac{q}{p+1} \times n$ will expreſs the Time required. In words thus;

Multiply the common Complement of the Lives by the Number of the Lives that are to drop, and divide the Product by the Number of all the Lives increaſed by Unity.

Thus, ſuppoſing 100 Lives, each of 40 Years of Age, it will be found that 5 of them will drop in about two Years and a Quarter.

But if we put t for the Time given, we ſhall have the four following Equations;

$$1^\circ.\ t = \frac{qn}{p+1}$$

$$2^\circ.\ q = \frac{\overline{p+1} \times t}{n}$$

$$3^\circ.\ p = \frac{nq-t}{t}$$

$$4^\circ.\ n = \frac{\overline{p+1} \times t}{q}$$

In which any three of the four Quantities n, p, q, t, being given, the fourth will be known.

This Speculation might be carried to any Number of unequal Lives: but my Deſign not being to perplex the Reader with too great Difficulties, I ſhall forbear at preſent to proſecute the thing any farther.

PROBLEM XXVI.

A, *who is* 30 *Years of Age, buys an Annuity of* 1 l. *for a limited Time of his Life, ſuppoſe* 10 *Years, on Condition that if he dies before the Expiration of that Time, the Purchaſe Money is wholly to be loſt to his Heirs; to find the preſent Value of the Purchaſe, ſuppoſing Intereſt at* 5 per Cent.

SOLUTION.

Let n be the Complement of *A*'s Life, m the limited Number of Years, p the Difference of n and m; Q the Value of an Annuity of 1 *l.* certain for m Years, and V the Value of the Perpetuity: then the preſent Value of the Purchaſe will be $\frac{mV+p-\overline{V+1}\times Q}{n}$. In Words thus;

1°. *Multiply* V, *the Value of a Perpetuity, at the given Rate of Intereſt, by* m *the limited Number of Years, and reſerve the Product.*

2°. *To the ſame* V *add Unity, and take the Difference between their Sum and* p, *which is the Exceſs of the Complement of* A'*s Age above the limited Number of Years: multiply this Difference by* Q, *an Annuity certain for* m *Years, to get the ſecond Product.*

3°. *Let the* Sum *of theſe Products, if* p *is* greater *than* $V+1$; *and their* Difference, *if it is* leſſer, *be divided by* n, *the Complement of* A'*s Age; and the Quotient ſhall be the Value of the Purchaſe.*

As, in the Queſtion propoſed, where $n=56$, $m=10$, $p=46$, $Q=7.7212$, and $V=20$; the firſt Product (mV) is that of 20 by 10, or 200. And $p-\overline{V+1}$ being $46-21=25$, the ſecond Product is 25×7.7212, that is 193.0302. The two Products *added* (p being greater than $V+1$) make 393.0302: which divided by 56 quotes, for the Anſwer, 7.0184 Years Purchaſe.

Note, 1. When it happens that p is equal to $V+1$; as, Intereſt being at 5 *per Cent*, if the Difference of n and m is 21; the ſecond Product $p-\overline{V+1}\times Q$ vaniſhing, the Anſwer is ſimply $\frac{mV}{n}$.

2. If $m = n$, or $p = 0$, ſeeing V is equal to $\frac{1}{r-1}$, the Expreſſion will be changed into $\frac{1}{r-1} - \frac{rQ}{n \times \overline{r-1}}$; which coincides with the Solution of *Prob.* I: Q repreſenting now the ſame Thing as P did in that Problem.

3. By this Propoſition, ſome uſeful Queſtions concerning *Inſurances* may be reſolved.

Suppoſe *A*, at 30 Years of Age, aſſigns over to *B* an Annuity of 1000 *l* a Year, limited to 10 Years, and depending likewiſe upon *A*'s Life: then, by the foregoing Solution, *A* ought to receive for it only 7018 *l.* 8 *s.* Intereſt being at 5 *per Cent.* But if *B* wants that the Annuity ſhould ſtand clear of all Riſques, he muſt pay for it the Value *certain*, which is 7721 *l.* 4 *s.* and *A* ought to have his Life inſured for 702 *l.* 16 *s.* the juſt Price of ſuch an Inſurance being *the Difference of the Values of the Annuity* certain, *and of the ſame Annuity* ſubject to the Contingency of the Annuitant's Life failing.

The ſame 702 *l.* 16 *s.* is likewiſe the Value of the Reverſion of this Annuity to a Perſon and his Heirs, who ſhould ſucceed to the Remainder of the 10 Years, upon *A*'s Deceaſe. *See* Prob. XXVIII.

It is evident by the foregoing Proceſs, that altho' the Queſtion there propoſed is particular, yet the Solution is general; which Method, often practiſed in my *Doctrine of Chances*, is of ſingular Uſe to fix the Reader's Imagination.

PROBLEM XXVII.

A *pays an Annuity of* 100 l. *during the Lives of* B *and* C, *each* 34 *Years of Age*; *to find what* A *ought to give in preſent Money to buy off the Life of* B, *ſuppoſing Intereſt at* 4 per Cent.

SOLUTION.

It will be found by our Tables that an Annuity upon a Life of 34 is worth 14.12 Years Purchaſe; and, by the Rules before delivered, that an Annuity upon the longeſt of the two Lives of *B* and *C* is worth 18.40: hence it is very plain, that, to buy off the Life of *B*, *A* muſt pay the Difference between 18.40 and 14.12, which being 4.28, it follows that *A* ought to pay 428 *l.*

In the ſame manner, if *A* were to pay an Annuity during the three Lives of *B*, *C*, *D*, whether of the ſame or different Ages, it would be eaſy to determine what *A* ought to pay to buy off one of the Lives of *B*, *C*, *D*, or any two of them, or to redeem the whole.

For, 1°. if the Life of *D* is to be bought off, then from the Value of the three Lives, ſubtract the Value of the two Lives of *B* and *C*, and the Remainder is what is to be given to buy off the Life of *D*.

2°. If the two Lives of *C* and *D* were to be bought off, then from the Value of the three Lives, ſubtract the Life of *B*, and the Remainder is what is to be given to buy off thoſe two Lives.

Laſtly, It is plain that to redeem the whole, the Value of the three Lives ought to be paid.

PROBLEM XXVIII.

A, *whoſe Life is worth* 14 *Years Purchaſe, ſuppoſing Intereſt at* 4 per Cent. *is to enjoy an Annuity of* 100 l. *during the Term of* 31 *Years*; B *and his Heirs have the Reverſion of it after the Deceaſe of* A *for the Term remaining*; *to find the Value of* B'*s Expectation.*

SOLUTION.

Since the Life of *A* is ſuppoſed to be worth 14 Years Purchaſe when Intereſt is at 4 *per Cent.* it follows from the Tables that *A* muſt be about 35 Years of Age, therefore find, by the twenty-ſixth Propoſition, the Value of an Annuity on a Life of 35, to continue the limited Time of 31 Years; let that Value be ſubtracted from the Value of an Annuity certain, to continue 31 Years; and the Remainder will be the Value of the Reverſion.

PROBLEM XXIX.

A *is to have an Annuity of* 100 l. *for him and his Heirs after the failing of any one of the Lives* M, P, Q, *the firſt of which is worth* 13 *Years Purchaſe, the ſecond* 14, *and the third* 15; *to find the preſent Value of his Expectation, Intereſt of Money being ſuppoſed at* 4 per Cent.

SOLUTION.

By the Example to Prob. III. it appears, that an Annuity upon the above 3 joint Lives is worth 7.41 Years Purchaſe; let this be ſuppoſed $=R$, and let ſ repreſent the preſent Value of a Perpetuity of 100 *l.* which in this Caſe is 2500 *l.* then the preſent Expectation of *A* will be worth $\overline{1-dR}\times ſ$. In Words thus;

Multiply the Value of the three joint Lives by the Intereſt of 1 l. *then ſubtracting that Product from Unity, let the Remainder be multiplied by the Value of the Perpetuity, and the Product will be the Expectation required.*

In this Caſe 7.41, multiplied by 0.04, produces 0.2964, and this Product ſubtracted from Unity, leaves 0.7036; now this Remainder being multiplied by 2500, produces 1759 *l.* the Expectation of *A*.

But if the Problem had been, that *A* ſhould not have the Annuity before the Failing of any two of thoſe Lives; from the Sum of all the joint Lives combined two and two, ſubtract the double Value of the three joint Lives, and let the Remainder be called *T*, then the Expectation of *A* will be worth $\overline{1-dT}\times ſ$; now, by the Rules before delivered, we ſhall find that the Sum of all the joint Lives combined two and two, is 29.06, from which ſubtracting the double of the three joint Lives, *viz.* 14.82, the Remainder is 14.24. Hence ſuppoſing $T=14.24$, then $\overline{1-dT}\times ſ$ will be found to be 1076 *l.* and this is the Value of *A*'s Expectation.

Laſtly, If *A* was not to have the Annuity before the Extinction of the three Lives, ſuppoſe the Value of the three Lives $=V$, then the Expectation of *A* would be worth $\overline{1-dV}\times ſ$, which in this Caſe is 465 *l.*

PROBLEM XXX.

To determine the Fines to be paid for renewing any Number of Years in a College-Leaſe of twenty; and alſo what Rate of Intereſt is made by a Purchaſer, who may happen to give an advanced Price for the ſame, upon Suppoſition that the Contractor is allowed 8 per Cent. *of his Money.*

Altho' the Problem here propoſed does not ſeem to relate to the Subject of this Book, yet as ſome uſeful Concluſions may be derived from the Solution of it, I have thought fit to inſert it in this Place.

Table of Fines.

1	0.2146	8	2.2821	15	5.8254
2	0.4463	9	2.6792	16	6.5060
3	0.6965	10	3.1081	17	7.2411
4	0.9666	11	3.5713	18	8.0349
5	1.2587	12	4.0715	19	8.9922
6	1.4133	13	4.6118	20	9.8181
7	1.9144	14	5.1953		

If a Purchaſer gives the Original Contractor 11 Years Purchaſe for his Leaſe of 20, he makes above $6\frac{1}{2}$ *per Cent.* of his Money.

If he gives 12 Years Purchaſe for the ſame, he makes above 5 *l.* 8 *s. per Cent.* of his Money.

If he gives 13 Years Purchaſe, he makes $4\frac{1}{2}$ *per Cent.* of his Money.

PROBLEM XXXI.

To determine the Fines to be paid for renewing any Number of Years in a College-Leaſe of One and Twenty; as alſo what Rate of Intereſt is made by a Purchaſer who may happen to give an advanced Price for the ſame, upon Suppoſition that the Contractor is allowed 8 per Cent. *of his Money.*

Table of Fines.

1	0.1987	8	2.1131	15	5.3940
2	0.4133	9	2.2808	16	6.0241
3	0.6450	10	2.8779	17	6.7047
4	0.8952	11	3.3068	18	7.4398
5	1.1653	12	3.7700	19	8.2336
6	1.4574	13	4.2702	20	9.0909
7	1.6120	14	4.8105	21	10.0168

He that gives 11 Years Purchaſe, inſtead of 10.0168 for renewing his Leaſe for 21 Years, makes 6 *l.* 16 *s. per Cent.* of his Money.

He who gives 12 Years Purchaſe for the ſame, makes very near 5 *l.* 16 *s. per Cent.* of his Money.

He who gives 13 Years Purchaſe for the ſame, makes a little more than 4 *l.* 16 *s. per Cent.* of his Money.

The Values of Annuities for Lives having been calculated, in this Book, upon a ſuppoſition that the Payments are made Yearly, and there being ſome Occaſions wherein it is ſtipulated that the Payments ſhould be made Half-Yearly, I have thought fit to add the two following Problems; whereby, 1°. *It is ſhewn what the Half-Yearly Payments ought to be, if the Price of the Purchaſe is preſerved.* 2°. *How the Price of the Purchaſe ought to be increaſed, if the Half-Yearly Payments are required to be the Half of the Yearly Payments.*

PROBLEM XXXII.

An Annuity being given, to find what Half-Yearly Payments will be equivalent to it, when Intereſt of Money is 4, 5, *or* 6 per Cent

SOLUTION.

Take Half of the Annuity, and from that Half ſubtract its 100th, or 80th, or 68th Part, according as the Intereſt is 4, 5, or 6 *per Cent.* and the Remainder will be the Value of the Half-Yearly Payments required; thus, if the Annuity was 100 *l.* the Half-Yearly Payments would reſpectively be 49 *l.* 10 *s.* 49 *l.* 7 *s.* 6 *d.* 49 *l.* 5 *s.* 3 *d.* nearly.

PROBLEM XXXIII.

The preſent Value of an Annuity being given, to find how much this preſent Value ought to be increaſed, when it is required that the Payments ſhall be Half-Yearly, and alſo one Half of the Yearly Payments, when Intereſt is at 4, 5, *or* 6 per Cent.

SOLUTION.

To the preſent Value of the Annuity add reſpectively its 99th, 79th, or 67th, and the Sums will be the Values increaſed.

As there are ſome Perſons who may be deſirous to ſee a general Solution of the two laſt Problems, I have thought fit to add what follows.

In the firſt of the two laſt Problems, let A be the Yearly Payments agreed on, and B the Half-Yearly Payments required, r the Yearly Rate of Intereſt, then $B = \frac{r^{\frac{1}{2}}-1}{r-1} \times A$. In the ſecond, let M be the preſent Value of the Yearly Payments, P the preſent Value of thoſe that are to be Half-Yearly, then $P = \frac{\frac{1}{2} \times \overline{r-1}}{r^{\frac{1}{2}}-1} \times M$.

TABLE I.

The preſent Value of an annuity of one pound, for any Number of Years not exceeding 100, *Intereſt at* 3 per Cent.

Years.	Value.	Years.	Value.	Years.	Value.	Years.	Value.
1	0.9709	26	17.8768	51	25.9512	76	29.8076
2	1.9135	27	18.3270	52	26.1662	77	29.9103
3	2.8286	28	18.7641	53	26.3750	78	30.0100
4	3.7170	29	19.1884	54	26.5777	79	30.1068
5	4.5797	30	19.6004	55	26.7744	80	30.2008
6	5.4172	31	20.0004	56	26.9655	81	30.2920
7	6.2303	32	20.3887	57	27.1509	82	30.3806
8	7.0197	33	20.7658	58	27.3310	83	30.4666
9	7.7861	34	21.1318	59	27.5058	84	30.5501
10	8.5302	35	21.4872	60	27.6756	85	30.6311
11	9.2526	36	21.8323	61	27.8404	86	30.7099
12	9.9540	37	22.1672	62	28.0003	87	30.7863
13	10.6350	38	22.4925	63	28.1557	88	30.8605
14	11.2961	39	22.8082	64	28.3065	89	30.9325
15	11.9379	40	23.1148	65	28.4529	90	31.0024
16	12.5611	41	23.4124	66	28.5950	91	31.0703
17	13.1611	42	23.7014	67	28.7330	92	31.1362
18	13.7535	43	23.9819	68	28.8670	93	31.2001
19	14.3238	44	24.2543	69	28.9971	94	31.2622
20	14.8775	45	24.5187	70	29.1234	95	31.3224
21	15.4150	46	24.7754	71	29.2460	96	31.3809
22	15.9369	47	25.0247	72	29.3651	97	31.4377
23	16.4436	48	25.2667	73	29.4807	98	31.4928
24	16.9355	49	25.5017	74	29.5929	99	31.5463
25	17.4131	50	25.7298	75	29.7018	100	31.5984

The Value of the Perpetuity is $33\frac{1}{3}$ *Years Purchaſe.*

TABLE

TABLE II.

The preſent Value of an Annuity of one Pound, to continue ſo long as a Life of a given Age is in being, Intereſt being eſtimated at 3 per Cent.

Age.	Value	Age.	Value	Age.	Value	Age.	Value
1	15.05	26	17.50	51	12.26	76	4.05
2	16.62	27	17.33	52	12.00	77	3.63
3	17.83	28	17.16	53	11.73	78	3.21
4	18.46	29	16.98	54	11.46	79	2.78
5	18.90	30	16.80	55	11.18	80	2.34
6	19.33	31	16.62	56	10.90	81	1.89
7	19.60	32	16.44	57	10.61	82	1.43
8	19.74	33	16.25	58	10.32	83	0.96
9	19.87	34	16.06	59	10.03	84	0.49
10	19.87	35	15.86	60	9.73	85	0.00
11	19.74	36	15.67	61	9.42	86	0.00
12	19.60	37	15.46	62	9.11		
13	19.47	38	15.26	63	8.79		
14	19.33	39	15.05	64	8.46		
15	19.19	40	14.84	65	8.13		
16	19.05	41	14.63	66	7.79		
17	18.90	42	14.41	67	7.45		
18	18.76	43	14.19	68	7.10		
19	18.61	44	13.96	69	6.75		
20	18.46	45	13.73	70	6.38		
21	18.30	46	13.49	71	6.01		
22	18.15	47	13.25	72	5.63		
23	17.99	48	13.01	73	5.25		
24	17.83	49	12.76	74	4.85		
25	17.66	50	12.51	75	4.45		

TABLE III.

The preſent Value of an Annuity of one Pound, for any Number of Years not exceeding 100, *Intereſt at* $3\frac{1}{2}$ per Cent.

Years.	Value.	Years.	Value.	Years.	Value.	Years.	Value.
1	0.9662	26	16.8904	51	23.6286	76	26.4799
2	1.8997	27	17.2854	52	23.7958	77	26.5506
3	2.8016	28	17.6670	53	23.9573	78	26.6190
4	3.6731	29	18.0358	54	24.1133	79	26.6850
5	4.5151	30	18.3920	55	24.2641	80	26.7488
6	5.3286	31	18.7363	56	24.4097	81	26.8104
7	6.1145	32	19.0689	57	24.5504	82	26.8700
8	6.8740	33	19.3902	58	24.6864	83	26.9275
9	7.6077	34	19.7007	59	24.8178	84	26.9831
10	8.3166	35	20.0007	60	24.9447	85	27.0368
11	9.0015	36	20.2905	61	25.0674	86	27.0887
12	9.6633	37	20.5705	62	25.1859	87	27.1388
13	10.3027	38	20.8411	63	25.3004	88	27.1873
14	10.9205	39	21.1025	64	25.4110	89	27.2341
15	11.5174	40	21.3551	65	25.5178	90	27.2793
16	12.0941	41	21.5991	66	25.6211	91	27.3230
17	12.6513	42	21.8349	67	25.7209	92	27.3652
18	13.1897	43	22.0627	68	25.8173	93	27.4060
19	13.7098	44	22.2828	69	25.9104	94	27.4454
20	14.2124	45	22.4955	70	26.0004	95	27.4835
21	14.6980	46	22.7009	71	26.0873	96	27.5203
22	15.1671	47	22.8994	72	26.1713	97	27.5558
23	15.6204	48	23.0912	73	26.2525	98	27.5902
24	16.0584	49	23.2766	74	26.3309	99	27.6234
25	16.4815	50	23.4556	75	26.4067	100	27.6554

The Value of the Perpetuity is $28\frac{1}{7}$ *Years Purchaſe.*

TABLE IV.

The preſent Value of an Annuity of one Pound, ſo long as a Life of a given Age is in being, Intereſt being eſtimated at $3\frac{1}{2}$ per Cent.

Age.	Value	Age.	Value	Age.	Value	Age.	Value
1	14.16	26	16.28	51	11.69	76	3.98
2	15.53	27	16.13	52	11.45	77	3.57
3	16.56	28	15.98	53	11.20	78	3.16
4	17.09	29	15.83	54	10.95	79	2.74
5	17.46	30	15.68	55	10.69	80	2.31
6	17.82	31	15.53	56	10.44	81	1.87
7	18.05	32	15.37	57	10.18	82	1.42
8	18.16	33	15.21	58	9.91	83	0.95
9	18.27	34	15.05	59	9.64	84	0.48
10	18.27	35	14.89	60	9.36	85	0.00
11	18.16	36	14.71	61	9.08	86	0.00
12	18.05	37	14.52	62	8.79		
13	17.94	38	14.34	63	8.49		
14	17.82	39	14.16	64	8.19		
15	17.71	40	13.98	65	7.88		
16	17.59	41	13.79	66	7.56		
17	17.46	42	13.59	67	7.24		
18	17.33	43	13.40	68	6.91		
19	17.21	44	13.20	69	6.57		
20	17.09	45	12.99	70	6.22		
21	16.96	46	12.78	71	5.87		
22	16.83	47	12.57	72	5.51		
23	16.69	48	12.36	73	5.14		
24	16.56	49	12.14	74	4.77		
25	16.42	50	11.92	75	4.38		

TABLE

TABLE V.

The preſent Value of an Annuity of one Pound, for any Number of Years not exceeding 100, *Intereſt at* 4 per Cent.

Years.	Value.	Years.	Value.	Years.	Value.	Years.	Value.
1	0.9615	26	15.9827	51	21.6714	76	23.7311
2	1.8860	27	16.3295	52	21.7475	77	23.7799
3	2.7750	28	16.6630	53	21.8726	78	23.8268
4	3.6298	29	16.9837	54	21.9929	79	23.8720
5	4.4518	30	17.2920	55	22.1086	80	23.9153
6	5.2421	31	17.5884	56	22.2198	81	23.9571
7	6.0020	32	17.8735	57	22.3267	82	23.9972
8	6.7327	33	18.1476	58	22.4295	83	24.0357
9	7.4353	34	18.4111	59	22.5284	84	24.0728
10	8.1108	35	18.6646	60	22.6234	85	24.1085
11	8.7604	36	18.9082	61	22.7148	86	24.1428
12	9.3850	37	19.1425	62	22.8027	87	24.1757
13	9.9856	38	19.3678	63	22.8872	88	24.2074
14	10.5631	39	19.5844	64	22.9685	89	24.2379
15	11.1183	40	19.7927	65	23.0466	90	24.2672
16	11.6522	41	19.9930	66	23.1218	91	24.2954
17	12.1656	42	20.1856	67	23.1940	92	24.3225
18	12.6592	43	20.3707	68	23.2635	93	24.3486
19	13.1339	44	20.5488	69	23.3302	94	24.3736
20	13.5903	45	20.7200	70	23.3945	95	24.3977
21	14.0291	46	20.8846	71	23.4562	96	24.4209
22	14.4511	47	21.0429	72	23.5156	97	24.4431
23	14.8568	48	21.1951	73	23.5727	98	24.4646
24	15.2469	49	21.3414	74	23.6276	99	24.4851
25	15.6220	50	21.4821	75	23.6804	100	24.5049

TABLE

TABLE VI.

The preſent Value of an Annuity of one Pound, to continue ſo long as a Life of a given Age is in being, Intereſt being eſtimated at 4 per Cent.

Age.	Value	Age.	Value	Age.	Value	Age.	Value
1	13.36	26	15.19	51	11.13	76	3.91
2	14.54	27	15.06	52	10.92	77	3.52
3	15.43	28	14.94	53	10.70	78	3.11
4	15.89	29	14.81	54	10.47	79	2.70
5	16.21	30	14.68	55	10.24	80	2.28
6	16.50	31	14.54	56	10.01	81	1.85
7	16.64	32	14.41	57	9.77	82	1.40
8	16.79	33	14.27	58	9.52	83	0.95
9	16.88	34	14.12	59	9.27	84	0.48
10	16.88	35	13.98	60	9.01	85	0.00
11	16.79	36	13.82	61	8.75	86	0.00
12	16.64	37	13.67	62	8.48		
13	16.60	38	13.52	63	8.20		
14	16.50	39	13.36	64	7.92		
15	16.41	40	13.20	65	7.63		
16	16.31	41	13.02	66	7.33		
17	16.21	42	12.85	67	7.02		
18	16.10	43	12.68	68	6.71		
19	15.99	44	12.50	69	6.39		
20	15.89	45	12.32	70	6.06		
21	15.78	46	12.13	71	5.72		
22	15.67	47	11.94	72	5.38		
23	15.55	48	11.74	73	5.02		
24	15.43	49	11.54	74	4.66		
25	15.31	50	11.34	75	4.29		

TABLE VII.

The preſent Value of an Annuity of one Pound, for any number of Years not exceeding 100, *Intereſt at* 5 per Cent.

Years.	Value.	Years.	Value.	Years.	Value.	Years.	Value.
1	0.9523	26	14.3751	51	18.3389	76	19.5094
2	1.8594	27	14.6430	52	18.4180	77	19.5328
3	2.7232	28	14.8981	53	18.4934	78	19.5550
4	3.5459	29	15.1410	54	18.5651	79	19.5762
5	4.3294	30	15.3724	55	18.6334	80	19.5964
6	5.0756	31	15.5928	56	18.6985	81	19.6156
7	5.7863	32	15.8026	57	18.7605	82	19.6339
8	6.4632	33	16.0025	58	18.8195	83	19.6514
9	7.1078	34	16.1929	59	18.8757	84	19.6680
10	7.7212	35	16.3741	60	18.9292	85	19.6838
11	8.3064	36	16.5468	61	18.9802	86	19.6988
12	8.8632	37	16.7112	62	19.0288	87	19.7132
13	9.3935	38	16.8678	63	19.0750	88	19.7268
14	9.8986	39	17.0170	64	19.1191	89	19.7398
15	10.3796	40	17.1590	65	19.1610	90	19.7522
16	10.8377	41	17.2943	66	19.2010	91	19.7640
17	11.2740	42	17.4232	67	19.2390	92	19.7752
18	11.6895	43	17.5459	68	19.2753	93	19.7859
19	12.0853	44	17.6627	69	19.3098	94	19.7961
20	12.4622	45	17.7740	70	19.3426	95	19.8058
21	12.8211	46	17.8800	71	19.3739	96	19.8151
22	13.1630	47	17.9810	72	19.4037	97	19.8239
23	13.4885	48	18.0771	73	19.4321	98	19.8323
24	13.7986	49	18.1687	74	19.4592	99	19.8403
25	14.0939	50	18.2559	75	19.4849	100	19.8479

TABLE VIII.

The preſent Value of an Annuity of one Pound, to continue ſo long as a Life of a given Age is in being, Intereſt at 5 per Cent.

Age.	Value	Age.	Value	Age.	Value	Age.	Value
1	11.96	26	13.37	51	10.17	76	3.78
2	12.88	27	13.28	52	9.99	77	3.41
3	13.55	28	13.18	53	9.82	78	3.03
4	13.89	29	13.09	54	9.63	79	2.64
5	14.12	30	12.99	55	9.44	80	2.23
6	14.34	31	12.88	56	9.24	81	1.81
7	14.47	32	12.78	57	9.04	82	1.38
8	14.53	33	12.67	58	8.83	83	0.94
9	14.60	34	12.56	59	8.61	84	0.47
10	14.60	35	12.45	60	8.39	85	0.00
11	14.53	36	12.33	61	8.16	86	0.00
12	14.47	37	12.21	62	7.93		
13	14.41	38	12.09	63	7.68		
14	14.34	39	11.96	64	7.43		
15	14.27	40	11.83	65	7.18		
16	14.20	41	11.70	66	6.91		
17	14.12	42	11.57	67	6.64		
18	14.05	43	11.43	68	6.36		
19	13.97	44	11.29	69	6.97		
20	13.89	45	11.14	70	5.77		
21	13.81	46	10.99	71	5.47		
22	13.72	47	10.84	72	5.15		
23	13.64	48	10.68	73	4.82		
24	13.55	49	10.51	74	4.49		
25	13.46	50	10.35	75	4.14		

TABLE IX.

The preſent Value of an Annuity of one Pound, for any Number of Years not exceeding 100, *Intereſt at* 6 per Cent.

Years.	Value.	Years.	Value.	Years.	Value.	Years.	Value.
1	0.9433	26	13.0031	51	15.8130	76	16.4677
2	1.8333	27	13.2105	52	15.8613	77	16.4790
3	2.6730	28	13.4061	53	15.9069	78	16.4896
4	3.4651	29	13.5907	54	15.9499	79	16.4996
5	4.2123	30	13.7648	55	15.9905	80	16.5091
6	4.9173	31	13.9290	56	16.0288	81	16.5180
7	5.5823	32	14.0840	57	16.0649	82	16.5264
8	6.2097	33	14.2302	58	16.0989	83	16.5343
9	6.8016	34	14.3681	59	16.1311	84	16.5418
10	7.3600	35	14.4982	60	16.1614	85	16.5489
11	7.8868	36	14.6209	61	16.1900	86	16.5556
12	8.3838	37	14.7367	62	16.2170	87	16.5618
13	8.8526	38	14.8460	63	16.2424	88	16.5678
14	9.2949	39	14.9490	64	16.2664	89	16.5734
15	9.7122	40	15.0462	65	16.2891	90	16.5786
16	10.1058	41	15.1380	66	16.3104	91	16.5836
17	10.4772	42	15.2245	67	16.3306	92	16.5883
18	10.8276	43	15.3061	68	16.3496	93	16.5928
19	11.1581	44	15.3831	69	16.3676	94	16.5969
20	11.4699	45	15.4558	70	16.3845	95	16.6009
21	11.7640	46	15.5243	71	16.4005	96	16.6046
22	12.0415	47	15.5890	72	16.4155	97	16.6081
23	12.3033	48	15.6500	73	16.4297	98	16.6114
24	12.5503	49	15.7075	74	16.4431	99	16.6145
25	12.7833	50	15.7618	75	16.4558	100	16.6175

TABLE X.

The preſent Value of an Annuity of one Pound, to continue ſo long as a Life of a given Age is in being, Intereſt being eſtimated at 6 per Cent.

Age.	Value	Age.	Value	Age.	Value	Age.	Value
1	10.80	26	11.90	51	9.34	76	3.66
2	11.53	27	11.83	52	9.20	77	3.31
3	12.04	28	11.76	53	9.04	78	2.95
4	12.30	29	11.68	54	8.90	79	2.57
5	12.47	30	11.61	55	8.72	80	2.18
6	12.63	31	11.53	56	8.56	81	1.78
7	12.74	32	11.45	57	8.38	82	1.36
8	12.79	33	11.36	58	8.20	83	0.92
9	12.84	34	11.60	59	8.02	84	0.77
10	12.84	35	11.18	60	7.83	85	0.00
11	12.79	36	11.09	61	7.63	86	0.00
12	12.74	37	11.00	62	7.42		
13	12.69	38	10.90	63	7.21		
14	12.63	39	10.80	64	7.00		
15	12.58	40	10.70	65	6.77		
16	12.53	41	10.60	66	6.53		
17	12.47	42	10.50	67	6.22		
18	12.41	43	10.37	68	6.03		
19	12.36	44	10.26	69	5.77		
20	12.30	45	10.14	70	5.50		
21	12.23	46	10.02	71	5.22		
22	12.17	47	9.90	72	4.93		
23	12.11	48	9.76	73	4.63		
24	12.04	49	9.63	74	4.32		
25	11.97	50	9.49	75	4.00		

Note; The 1*ſt*, 3*d*, 5*th*, 7*th* and 9*th* Tables ſerve likewiſe to reſolve the Queſtions concerning *Compound Intereſt:* as

I.

To find the preſent Value of 1000 l. *payable* 7 *Years hence, at* $3\frac{1}{2}$ per Cent. From the preſent Value of an Annuity of 1 *l.* certain for 7 Years, which, in *Tab.* III. is 6.1145, I ſubtract the like Value for 6 Years, which is 5.3286; and the Remainder .7859 is the Value of the 7*th* Year's Rent, or of 1 *l.* payable after 7 Years; which multiplied by 1000 gives the Anſwer 785 *l.* 18*ſh.*

II.

If it is aſked, *what will be the* Amount *of the Sum* S *in* 7 *Years at* $3\frac{1}{2}$ per Cent? Having found .7859 as above, 'tis plain the *Amount* will be $\frac{S}{.7859}$.

III.

If the Queſtion is, *In what time a Sum* S *will be doubled, tripled, or increaſed in any given* Ratio *at* 3, $3\frac{1}{2}$, &c. per Cent. I take, in the proper *Table*, two contiguous Numbers whoſe Difference is neareſt the Reciprocal of the *Ratio* given, as $\frac{1}{2}$, $\frac{1}{3}$, &c. And the Year againſt the higher number is the Anſwer.

Thus in *Tab.* I. againſt the Years 22, 23, ſtand the Numbers 15.9369 and 16.4436; whoſe Difference .5067 being a little more than .5, or $\frac{1}{2}$, ſhews that in 23 Years, a Sum *S* will be a little leſs than doubled, at 3 *per Cent.* Compound Intereſt. And againſt the Years 36 and 37 are 21.8323, and 22.1672; the Difference whereof being .3349, nearly $\frac{1}{3}$, ſhews that in 14 Years more it will be almoſt tripled.

If more exactneſs is required; take the adjoining Difference whoſe Error is contrary to that of the Difference found; and thence compute the proportional part to be added or ſubtracted thus, in the laſt of theſe Examples, the Difference between the Years 37 and 38 is .3252, which wants .0081 of .3333 $\left(=\frac{1}{3}\right)$, as the other Difference .3349 exceeded it by .0016. The 38*th* Year is therefore to be divided in the Ratio of 16 to 81; that is $\frac{16}{97}$ of a Year, or about 2 Months, is to be added to the 37 Years.

IV.

To find at what Rate of Intereſt I ought to lay out a Sum S, *ſo as it may encreaſe* $\frac{1}{3}$ *for Inſtance, or become* $\frac{4}{3}$ S *in* 7 *Years.* Here the Fraction I am to look for among the Differences is $\frac{3}{4}$, or the Decimal .75; which is not to be found in *Tab.* I. or III, till after the limited Time of 7 Years. But in *Tab.* V, the Numbers againſt 6 and 7 Years give the Difference .7599; and the Rate is 4 *per Cent.* nearly.

To find how *nearly*; we may proceed as under the foregoing Rule. Take the Difference between 6 and 7 Years in *Tab.* VII. for 5 *per Cent.*; which being .7107, wanting .0393 of .75, as .7599 exceeded it by .0099; divide Unity in the Ratio of 99 to 393, that is of 33 to 131, and the leſſer Part added to 4 *per Cent.* gives the Rate ſought, $4\frac{33}{104}$, or $4\frac{1}{5}$.

PART

P A R T II.

Containing the Demonſtrations of ſome of the principal Propoſitions in the foregoing Treatiſe.

CHAPTER I.

I Obſerved formerly, that upon Suppoſition that the Decrements of Life were in Arithmetic Progreſſion, the Concluſions derived from thence would very little vary from thoſe, that could be deduced from the Table of Obſervations made at *Breſlaw*, concerning the Mortality of Mankind; which Table was about fifty Years ago inſerted by Dr. *Halley* in the *Philoſophical Tranſactions*, together with ſome Calculations concerning the Values of Lives according to a given Age.

Upon the foregoing Principle, I ſuppoſed that if *n* repreſented the Complement of Life, the Probabilities of living 1, 2, 3, 4, 5, *&c.* Years, would be expreſſed by the following Series, $\frac{n-1}{n}$, $\frac{n-2}{n}$, $\frac{n-3}{n}$, $\frac{n-4}{n}$, $\frac{n-5}{n}$, *&c.* and conſequently that the Value of a Life, whoſe Complement is n, would be expreſſed by the Series $\frac{n-1}{nr}+\frac{n-2}{nrr}+\frac{n-3}{nr^3}+\frac{n-4}{nr^4}+\frac{n-5}{nr^5}$, *&c.* the Sum of which I have aſſerted in *Problem* I. to be $\frac{1-\frac{r}{n}P}{r-1}$, where the Signification of the Quantities P and r is explained.

As the Reaſonings that led me to that general Expreſſion, require ſomething more than an ordinary Skill in the Doctrine of Series, I ſhall forbear to mention them in this Place; and content myſelf with pointing out to the Reader a Method, whereby he may ſatisfy himſelf of the Truth of that Theorem, provided he underſtand ſo much of a Series, as to be able to ſum up a Geometric Progreſſion.

DEMONSTRATION.

$$P=\frac{1}{r}+\frac{1}{rr}+\frac{1}{r^3}+\frac{1}{r^4}+\frac{1}{r^5}\ldots\frac{1}{r^n}.$$

Therefore,

$$rP=1+\frac{1}{r}+\frac{1}{rr}+\frac{1}{r^3}+\frac{1}{r^4}\ldots+\frac{1}{r^{n-1}}.$$

And

$$\frac{rP}{n}=\frac{1}{n}+\frac{1}{nr}+\frac{1}{nrr}+\frac{1}{nr^3}+\frac{1}{nr4}\ldots+\frac{1}{nr^{n-1}}.$$

Therefore,

$$1-\frac{rP}{n}=\frac{n-1}{n}-\frac{1}{nr}-\frac{1}{nrr}-\frac{1}{nr^3}-\frac{1}{nr^4}-\ldots\frac{1}{nr^{n-1}}.$$

But this is to be divided by $r-1$, or multiplied by $\frac{1}{r-1}=\frac{1}{r}+\frac{1}{rr}+\frac{1}{r^3}+\frac{1}{r^4}+\frac{1}{r^5}+\frac{1}{r6}$, &c.

Then multiplying actually thoſe two Series's together, the Product will be found to be

$$\begin{array}{lllllll}
\frac{n-1}{nr} & -\frac{1}{nrr} & -\frac{1}{nr^3} & -\frac{1}{nr^4} & -\frac{1}{nr^5} & -\frac{1}{nr^6} & \&c. \\
 & +\frac{n-1}{nrr} & -\frac{1}{nr^3} & -\frac{1}{nr^4} & -\frac{1}{nr^5} & -\frac{1}{nr^6} & \&c. \\
 & & +\frac{n-1}{nr^3} & -\frac{1}{nr^4} & -\frac{1}{nr^5} & -\frac{1}{nr^6} & \&c. \\
 & & & +\frac{n-1}{nr^4} & -\frac{1}{nr^5} & -\frac{1}{nr^6} & \&c. \\
 & & & & +\frac{n-1}{nr^5} & -\frac{1}{nr^6} & \&c. \\
 & & & & & +\frac{n-1}{nr^6} & \&c.
\end{array}$$

And adding the Terms of the perpendicular Columns together, we ſhall have $\frac{n-1}{nr}+\frac{n-2}{nrr}+\frac{n-3}{nr^3}+\frac{n-4}{nr^4}+\frac{n-5}{nr^5}+\frac{n-6}{nr^6}$ &c.

which conſequently is equal to $\frac{1-\frac{r}{n}P}{r-1}$: which was to be demonſtrated.

If it be required that upon the Failing of a Life, ſuch Part of the Annuity ſhould be paid, as may be proportional to the Time elapſed from the Beginning of the laſt Year, to the Time of the Life's failing, then the Value of the Life will be $\frac{1}{r-1}-\frac{1}{an}P$, wherein a repreſents the hyperbolic Logarithm of the Rate of Intereſt.

But becauſe there are no Tables printed of hyperbolic Logarithms, and that the Reduction of a common Logarithm to an hyperbolic is ſomewhat laborious, it will be ſufficient here to ſet down the hyperbolic Logarithms of 1.03, 1.035, 1.04, 1.05, 1.06, which are reſpectively, 0.02956, 0.0344, 0.03922, 0.04879, 0.05825, or $\frac{1}{35}$, $\frac{1}{31}$, $\frac{2}{51}$, $\frac{2}{41}$, $\frac{6}{103}$ nearly.

CHAPTER II.

Explaining the Rules of combined Lives.

Suppoſing a fictitious Life, whoſe Number of Chances to continue in being from Year to Year, are conſtantly equal to a, and the Number of Chances for failing are conſtantly equal to b, ſo that the Odds of its continuing during the Space of any one Year, be to its failing in the ſame Interval of Time conſtantly as a to b, the Value of an Annuity upon ſuch a Life would be eaſily found.

For, if we make $a+b=s$, the Probabilities of living 1, 2, 3, 4, 5, *&c.* Years would be repreſented by the Series $\frac{a}{s}$, $\frac{aa}{ss}$, $\frac{a^3}{s^3}$, $\frac{a^4}{s^4}$, $\frac{a^5}{s^5}$, *&c.* continued to Eternity; and conſequently the Value of an Annuity upon ſuch Life would be expreſſed by this new Series $\frac{a}{sr}+\frac{aa}{ssrr}+\frac{a^3}{s^3r^3}+\frac{a^4}{s^4r^4}$ *&c.* which being a geometric Progreſſion perpetually decreaſing, the Sum of it will be found to be $\frac{a}{sr-a}$: thus, if a ſtands for 21, and b for 1, and alſo r for 1.05, the Value of ſuch Life would be ten Years Purchaſe.

From theſe Premiſes the following Corollaries may be drawn:

COROLLARY I.

An Annuity upon a fictitious Life being given, the Probability of its continuing one Year in being is alſo given; for let the Value be $=M$, then $\frac{a}{s}=\frac{Mr}{M+1}$.

COROLLARY II.

If a Life, whoſe Value is deduced from our Tables is found to be worth 10 Years Purchaſe, then ſuch Life is equivalent to a fictitious Life, whoſe Number of Chances for continuing one Year, is to the Number of Chances for its failing in that Year, as 21 to 1.

COROLLARY III.

Wherefore having taken the Value of a Life from our Tables, or calculated it according to the Rules preſcribed; we may transfer the Value of that Life to that of a fictitious Life, and find the Number of Chances it would have for continuing or failing Yearly.

COROLLARY IV.

And the Combination of two or more *real Lives* will be very near the ſame as the Combination of ſo many correſponding *fictitious Lives*; and therefore an Annuity granted upon one or more real Lives, is nearly of the ſame Value as an Annuity upon a fictitious Life.

Theſe things being premiſed, it will not be difficult to determine the Value of an Annuity upon two or three, or as many joint Lives as may be aſſigned.

For let x repreſent the Probability of one Life's continuing from Year to Year, and y the Probability of another Life's continuing the ſame Time; then according to the Principles of the Doctrine of Chances, the Terms

$$xy,\ xxyy,\ x^3 y^3,\ x^4 y^4,\ x^5 y^5,\ \&c.$$

will reſpectively repreſent the Probabilities of continuing together, 1, 2, 3, 4, 5, &c. Years; and the Value of an Annuity upon the two joint Lives, will be $\frac{xy}{r} + \frac{xxyy}{rr} + \frac{x^3y^3}{r^3} + \frac{x^4y^4}{r^4} + \frac{x^5y^5}{r^5}$ &c. which being a Geometrical Progreſſion perpetually decreaſing, the Sum of it will be found to be $\frac{xy}{r-xy}$: let now M be put for the Value of the firſt Life, and P for the Value of the ſecond, then by our firſt Corollary it appears that $x = \frac{Mr}{M+1}$, and $y = \frac{Pr}{P+1}$; and therefore having written theſe Values of x and y in the Expreſſion $\frac{xy}{r-xy}$, which is the Value of the two joint Lives, it will be changed into $\frac{MPr}{\overline{M+1} \times \overline{P+1} - MPr}$: which is the ſame Theorem that I had given in my firſt Edition.

It is true that in the Solution of *Prob.* II. I have given a Theorem which ſeems very different from this; making the Value of the joint Lives to be $\frac{MP}{M+P-dMP}$, wherein d repreſents the Intereſt of 1 *l.* and yet I may aſſure the Reader, that this laſt Expreſſion is originally derived from the firſt; and that whether one or the other is uſed, the Concluſions will very little differ: but the firſt Theorem is better adapted to Annuities paid in Money, it being cuſtomary that the laſt Payment, whether it be Yearly or Half-Yearly, is loſt to the Purchaſer; whereas the ſecond Theorem is better fitted to Annuities paid by a Grant of Lands, whereby the Purchaſer makes Intereſt of his Money to the laſt Moment of his Life: for which Reaſon I have choſe to uſe the laſt Expreſſion in my Book.

By following the ſame Method of Inveſtigation, we ſhall find that if M, P, Q, denote three ſingle Lives, an Annuity upon thoſe joint Lives will be $\frac{MPQrr}{\overline{M+1}\times\overline{P+1}\times\overline{Q+1}-MPQrr}$, in the Caſe of Annuities payable in Money; or $\frac{MPQ}{MP+MQ+PQ-2dMQ}$, in the Caſe of Annuities paid by a Grant of Lands.

CHAPTER III.

Containing the Demonſtration of the Rules given in Problems 4th and 5th, for determining the Value of longeſt Life.

Let x and y repreſent the reſpective Probabilities which two Lives have of continuing one Year in being, therefore $1-x$ is the Probability of the firſt Life's failing in one Year, and $1-y$ the Probability of the ſecond Life's failing in one Year: Therefore multiplying theſe two Probabilities together, the Product $1-x-y+xy$ will repreſent the Probability of the two Lives failing in one Year; and if this be ſubtracted from Unity, the Remainder $x+y-xy$ will expreſs the Probability of one at leaſt of the two Lives outliving one Year: which is ſufficient for eſtabliſhing the firſt Year's Rent.

And, for the ſame Reaſon $xx+yy-xxyy$ will expreſs the Probability of one at leaſt of the two Lives outliving two Years: which is ſufficient to eſtabliſh the ſecond Year's Rent.

From the two Steps we have taken, it plainly appears that the longeſt of two Lives is expreſſible by the three following Series;

$$\left.\begin{array}{l}\frac{x}{r}+\frac{xx}{rr}+\frac{x^3}{r^3}+\frac{x^4}{r^4}+\frac{x^5}{r^5}\\+\frac{y}{r}+\frac{yy}{rr}+\frac{y^3}{r^3}+\frac{y^4}{r^4}+\frac{y^5}{r^5}\\-\frac{xy}{r}-\frac{xxyy}{rr}-\frac{x^3y^3}{r^3}-\frac{x^4y^4}{r^4}-\frac{x^5y^5}{r^5}\end{array}\right\}\ \&c.$$

Whereof the firſt repreſents an Annuity upon the firſt Life, the ſecond an Annuity upon the ſecond Life, and the third an Annuity upon the two joint Lives; and therefore we may conclude that an Annuity upon the longeſt of two Lives, is the Difference between the Sum of the Values of the ſingle Lives, and the Value of the joint Lives: which have been expreſſed in *Problem* IV. by the Symbols $M+P-\overline{MP}$.

In the ſame manner it will be found that if x, y, z, repreſent the reſpective Probabilities of three Lives continuing one Year, then the Probability of their not failing all three in one Year will be expreſſed by $x+y+z-xy-xz-yz+xyz$; which is ſufficient to ground this Concluſion, that an Annuity upon the longeſt of three Lives, is the Sum of the ſingle Lives, *minus* the Sum of the joint Lives, *plus* the three joint Lives: which has been expreſſed by me, by the Symbols $M+P+Q-\overline{MP}-\overline{MQ}-\overline{PQ}+\overline{MPQ}$.

From the foregoing Concluſions, it is eaſily perceived how the Value of the longeſt of any Number of Lives ought to be determined; *viz.* by the Sum of the Values of the ſingle Lives, *minus* the Sum of the Values of all the joint Lives taken two and two, *plus* the Sum of all the joint Lives taken three and three, *minus* the Sum of all the joint Lives taken four and four, and ſo on by alternate Additions and Subtractions.

CHAPTER IV.

Containing the Demonſtrations of what has been ſaid concerning Reverſions, and the Value of one Life after one or more Lives.

1°. It plainly appears that the preſent Value of a Reverſion after one Life, is the Difference between the Perpetuity, and the Value of the Life in Poſſeſſion: Thus, if the Life in Poſſeſſion be worth 14 Years Purchaſe, and that I have the Reverſion after that Life, and have a mind to ſell it, I muſt have for it 11 Years purchaſe, which is the Difference between the Perpetuity 25, and 14 the Value of the Life, when Money is rated at 4 *per Cent.*

2°. It is evident that the Reverſion after two, three, or more Lives, is the Difference between the Perpetuity, and the longeſt of all the Lives.

But the Value of a Life after one or more Lives not being ſo obvious, I think it is proper to inſiſt upon it more largely: let x therefore repreſent the Probability of the Expectant's Life continuing one Year in being, and y the Probability of the ſecond Life's continuing alſo one Year in being, and therefore $1-y$ is the Probability of that ſecond Life's failing in that Year; from which it follows, according to the Doctrine of Chances, that the Probability of the firſt Life's continuing one Year, and of the ſecond's failing in that Year, is $x \times \overline{1-y}$, or $x-xy$; which is a ſufficient foundation for drawing the following Concluſion, *viz.* that the Value of the firſt Life after the ſecond is the Value of that firſt Life *minus* the Value of the two joint Lives: which I have expreſſed by the Symbols $M-\overline{MP}$.

In the ſame manner, if x, y, z, repreſent the reſpective Probabilities of three Lives continuing one Year, then $x \times \overline{1-y} \times \overline{1-z}$, will repreſent the Probability of the firſt Life's continuing one Year, and of the other two Lives failing in that Year; but the foregoing Expreſſion is brought, by actual Multiplication, to its Equivalent $x-xy-xz+xyz$; from whence can be deduced by meer Inſpection the Rule given in *Prob.* VIII. *viz.* that the preſent Value of the firſt Life's Expectation after the Failing of the other two, is $M-\overline{MP}-\overline{MQ}+\overline{MPQ}$.

CHAPTER V.

Containing the Demonſtration of what has been aſſerted in the Solution of the 10th and 29th Problems.

In the Solution of the 10th Problem, M''' denoting the preſent Value of an Annuity to continue ſo long as three Lives of the ſame Age ſubſiſt together, let us ſuppoſe that n denotes the Number of Years during which the Annuity will continue; then ſuppoſing r to expreſs the Rate of Intereſt, it is well known that the preſent Value of that Annuity will be $\dfrac{1-\frac{1}{r^n}}{r-1}$, wherefore we have the Equation $M'''=\dfrac{1-\frac{1}{r^n}}{r-1}$, or making $r-1=d$, $M'''=\dfrac{1-\frac{1}{r^n}}{d}$, from whence will be deduced $\dfrac{1}{r^n}=1-dM'''$, and conſequently $r^n=\dfrac{1}{1-dM'''}$. Now let us ſuppoſe that a Sum s is to be received to eternity at the equal Intervals of Time, denoted by n, and that we want to find the preſent Value of it; it is plain to thoſe who have made ſome Proficiency in Algebra, that $\frac{ſ}{r^n - 1}$ is the preſent Value of it, let us therefore in the room of r^n ſubſtitute its Value found before, *viz.* $\frac{1}{1 - dM'''}$, and then $r^n - 1$ will be found equal to $\frac{dM'''}{1 - aM'''}$, and conſequently $\frac{ſ}{r^n - 1} = \frac{1 - dM'''}{dM'''} \times ſ$: as in the Solution of *Prob.* X.

Now it will be eaſy to find n; for let us ſuppoſe $\frac{1}{1 - dM'''} = T$, then $r^n = T$, and therefore $n = \frac{log.\,T}{log.\,r}$.

The 29th Problem has ſome Affinity with the 10th; in the former it was required to know the preſent Value of a Sum $ſ$, payable at the Failing of any one of three equal Lives, but in the latter the three Lives are ſuppoſed unequal; but beſides, it is extended to two other Caſes, *viz.* to the preſent Value of a Sum $ſ$ to be paid after the Failing of any two of the Lives, as alſo to the preſent Value of a Sum $ſ$ to be paid after the Failing of the three Lives.

For in the firſt Caſe, let us imagine an Annuity to be paid as long as the three Lives are in being; or, which is the ſame thing, till one of the Lives fails; and let us ſuppoſe that R repreſents the Value of the three joint Lives; let us alſo ſuppoſe that n is the Number of Years after which this will happen, and that d is the Intereſt of 1 *l.* therefore $\frac{ſ}{r^n}$ is the preſent Value of the Sum $ſ$ to be then paid; but $R = \frac{1 - \frac{1}{r^n}}{d}$, therefore $\frac{1}{r^n} = 1 - dR$, and therefore $\frac{ſ}{r^n} = \overline{1 - dR} \times ſ$.

But the ſecond Caſe has ſomething more of Difficulty, and therefore I ſhall enlarge a little more upon it: let us imagine now that there is an Annuity to continue not only as long as the three equal Lives are in being, but as long as any two of the ſaid Lives are in being; now in order to find the preſent Value of the ſaid Annuity, let us ſuppoſe that x, y, z, repreſent the reſpective Probabilities of the ſaid Lives continuing one Year. Therefore.

1°. xyz repreſents the Probability of their all outliving the Year.

2°. $xy \times \overline{1 - z}$, or $xy - xyz$ repreſents the Probability of the two firſt outliving the Year, and of the third failing in that Year.

3°.

3°. $xz\times\overline{1-y}$ or $xz-xyz$ repreſents the Probability of the firſt and third's outliving the Year, and of the ſecond's failing in that Year.

4°. $yx\times\overline{1-z}$, or $yz-xyz$ repreſents the Probability of the ſecond and third's outliving the Year, and of the firſt's failing in that Year.

Then adding thoſe ſeveral Products together, their Sum will be found equal to $xy+xz+yz-2xyz$, which is an Indication that the preſent Value of an Annuity to continue as long as two of the ſaid Lives are in being is $\overline{MP}+\overline{MQ}+\overline{PQ}-\overline{2MPQ}$, which we may ſuppoſe $=T$.

Let us now compare this with an Annuity certain to continue n Years, the Rate of Intereſt being ſuppoſed $=r$, and $r-1=d$, then we ſhall have the Equation $\frac{1-\frac{1}{r^n}}{d}=T$, from whence we ſhall find $\frac{1}{r^n}=1-dT$, and conſequently $\frac{s}{r^n}$, which is the preſent Value of the Expectation required, is $=\overline{1-dT}\times s$.

By the ſame Method of Proceſs, we may find the preſent Value of an Annuity to continue ſo long as any one of the three Lives in queſtion is ſubſiſting; for let x, y, z, repreſent the ſame things as before.

1°. xyz repreſents the Probability of the three Lives outliving the firſt Year.

2°. $xy+xz+yz-3xyz$ repreſents the Probability of two of them outliving the Year, and of the third's failing in that Year.

3°. $x\times\overline{1-y}\times\overline{1-z}$, or $x-xy-xz+xyz$ repreſents the Probability of the firſt Life's outliving the Year, and of the other two failing in that Year.

4°. $y\times\overline{1-x}\times\overline{1-z}$, or $y-xy-zy+xyz$ repreſents the Probability of the ſecond Life's outliving the Year, and of the other two failing in that Year.

5°. $z\times\overline{1-x}\times\overline{1-y}$, or $z-xz-yz+xyz$ repreſents the Probability of the third Life's outliving the Year, and of the other two failing in that Year.

Now the Sum of all this is $x+y+z-xy-xz-yz+xyz$; which is an Indication that the Value of an Annuity to continue as long as any one of three Lives is in being ought to be expreſſed by $M+P+Q-\overline{MP}-\overline{MQ}-\overline{PQ}+\overline{MPQ}$: and this laſt Caſe may be looked upon as a Confirmation of the Rule given in our 5th Problem.

CHAPTER VI.

Containing the Demonſtration of what has been ſaid concerning ſucceſſive Lives in the Solution of Prob. XIII.

What has been there ſaid amounts to this; The preſent Values of Annuities certain for any particular Number of Years being given, to find the preſent Value of an Annuity to continue as long as the Sum of thoſe Years.

Let us ſuppoſe that M repreſents the preſent Value of an Annuity to continue n Years, and that P repreſents the preſent Value of an Annuity to continue p Years; the firſt Queſtion is, how from theſe *Data* to find the preſent Value of an Annuity to continue $n+p$ Years, the Inveſtigation of which is as follows: let r be the Rate of Intereſt, and ſuppoſe $r-1$ which denotes the Intereſt of 1 *l.* $= d$; then, 1°. $M = \frac{1-\frac{1}{r^n}}{d}$, therefore $\frac{1}{r^n} = 1 - dM$; and for the ſame Reaſon $\frac{1}{r^p} = 1 - dP$. Therefore $\frac{1}{r^{n+p}} = \overline{1-dM} \times \overline{1-dP} = 1 - dM - dP + ddMP$. Let now $ſ$ be ſuppoſed to be the Value of the Annuity which is to continue $n+p$ Years, then $\frac{1}{r^{n+p}} = 1 - dſ$. Therefore $1 - dſ = 1 - dM - dP + ddMP$; then ſubtracting Unity on both Sides, dividing all by d, and changing the Signs, we ſhall have $ſ = M + P - dMP$.

2°. By the ſame Method of Proceſs, it will be eaſy to find that if M, P, Q, repreſent Annuities to continue for the reſpective Number of Years n, p, q, then the Value of an Annuity to continue $n+p+q$ Years will be $M + P + Q - dMP - dMQ - dPQ + ddMPQ$: the Continuation of which is obvious.

Let us now ſuppoſe that the Intervals n, p, q, are equal, then the Values M, P, Q, are alſo equal; in which Caſe, the foregoing Canon will be changed into this, $3M - 3dMM + d^2M^3$, or $\frac{3dM - 3ddMM + d^3M^3}{d}$: but if this Numerator be ſubtracted from Unity, the Remainder will be $1 - 3dM + 3ddMM - d^3M^3 = \overline{1-dM}|^3$; and ſubtracting this again from Unity, the original Numerator will be reſtored, and will be equivalent to $1 - \overline{1-dM}^3$, and conſequently, if M repreſents the Value of an Annuity to continue

tinue a certain Number of Years, then $\frac{1-\overline{1-dM}^{\,3}}{d}$ will repreſent the Value of an Annuity to continue three times as long.

And univerſally, if M ſtands for the Value of an Annuity to continue a certain Number of Years, then $\frac{1-\overline{1-dM}^{\,n}}{d}$ will repreſent the Value of an Annuity to continue n times as long.

And if n were infinite, I ſay that $\overline{1-dM}^{\,n}$ would be $=0$; from whence the Value would be $=\frac{1}{d}$ or $\frac{1}{r-1}$, which repreſents the Value of the Perpetuity.

But that there may remain no ſcruple about what we have aſſerted above, that in the Caſe of n being infinite, $\overline{1-dM}^{\,n}$ would vaniſh; I prove it thus, $\frac{1}{d} > M$, therefore $1 > dM$, therefore $1-dM$ is a Fraction leſs than Unity: now it is well known that a Fraction leſs than Unity being raiſed to an infinite Power, is nothing, and was therefore ſafely neglected.

CHAPTER VII.

Containing the Demonſtration of what has been aſſerted in the 32d and 33d Problems concerning half-yearly Payments; as alſo the Inveſtigation of ſome Theorems relating to that Subject.

It is well known that if an Annuity A is to continne n Years, the preſent Value of it is $\frac{A-\frac{A}{r^n}}{r-1}$; ſuppoſing r to repreſent the Rate of Intereſt; now to make a proper Application of this Theorem to half-yearly Payments, I look upon n as repreſenting indifferently the Number of Payments and the Number of Years; let us now ſuppoſe a half-yearly Rent B of the ſame preſent Value as the former, and to continue as long, then the Number of Payments in this Caſe will be $2n$, but the Rate of Intereſt, inſtead of being r, is now $r^{\frac{1}{2}}$, which being raiſed to the Power $2n$, will be r^n as before; for which Reaſon the preſent Value of the half-yearly Payments is $\frac{B-\frac{B}{r^n}}{r^{\frac{1}{2}}-1}$: but by Hypotheſis, the preſent Values of the yearly and half-yearly

Payments are the ſame; therefore $\frac{A-\frac{A}{r^n}}{r-1}=\frac{B-\frac{B}{r^n}}{r^{\frac{1}{2}}-1}$, and dividing both ſides of the Equation by $1-\frac{1}{r^n}$, we ſhall have $\frac{A}{r-1}=\frac{B}{r^{\frac{1}{2}}-1}$, from whence will be deduced $B=\frac{r^{\frac{1}{2}}-1}{r-1}\times A$: and in the ſame manner, if the Payments were to be made quarterly, then B would be $=\frac{r^{\frac{1}{4}}-1}{r-1}\times A$; and ſo on.

But if we ſuppoſe that a Rent ſhall be paid half-yearly, and that it ſhall be alſo one half of what would be given for an annual Rent, and that the two Rents ſhall be of the ſame Duration; then the preſent Values of the yearly and half-yearly Rents will be different: for let M and P be the preſent Values of the yearly and half-yearly Rents, then $M=\frac{A-\frac{A}{r^n}}{r-1}$, and $P=\frac{\frac{1}{2}A-\frac{\frac{1}{2}A}{r^n}}{r^{\frac{1}{2}}-1}$, and dividing both Values by $A-\frac{A}{r^n}$, we ſhall have $M, P :: \frac{1}{r-1}, \frac{\frac{1}{2}}{r^{\frac{1}{2}}-1}$; and conſequently $P=\frac{\frac{1}{2}\times\overline{r-1}}{r^{\frac{1}{2}}-1}\times M$.

The two laſt Problems bring to my Mind an Aſſertion which was maintained, about ſix Years ago, in a Pamphlet then publiſhed; which was that it would be of great Advantage to a Perſon who pays an Annuity, to diſcharge it by half-yearly Payments, each of one half the Annuity in Queſtion: the Reaſon of which was, that then the time of paying off the Principal would be conſiderably ſhortened. I had not the Curioſity to read the Author's Calculation, becauſe I thought it too long; ſince which Time I thought fit to examine the thing, and found that indeed the Time would be ſhortened, but not ſo conſiderably as the Author imagined: which to prove, I ſuppoſed a Principal of 2000 *l.* an Annuity of 100 *l.* and the Rate of Intereſt 1.04: in conſequence of which, I found that the Principal would be diſcharged in 41 Years; this being founded on the general Theorem $\frac{A-\frac{A}{r^n}}{r-1}=P$, in which A repreſents the Annuity, P the Principal, r the Rate of Intereſt, and n the Number of Years: now to apply this to the Caſe of half-yearly Payments, let us ſuppoſe that p denotes the Number of Years in which the Principal will be diſcharged; therefore $2p$ will be the Number of Payments, $\frac{1}{2}A$ the Annuity, and $r^{\frac{1}{2}}$ the Rate of Intereſt: which being reſpectively ſubſtituted in the Room of

n, A, r, we ſhall have now $\dfrac{\frac{1}{2}A - \dfrac{\frac{1}{2}A}{r^p}}{r^{\frac{1}{2}} - 1} = P$, but $r^{\frac{1}{2}} - 1 = 0.019804$, which being ſuppoſed $= m$, we ſhall have $\frac{1}{2}A - \dfrac{\frac{1}{2}A}{r^p} = m\,P$, and $\dfrac{\frac{1}{2}A}{r^p} = \frac{1}{2}A - mP$, or $\dfrac{50}{r^p} = 10.392$; therefore $\dfrac{r^p}{50} = \dfrac{1}{10.392}$, or $r^p = \dfrac{50}{10.392}$, and p *log.* $r =$ *log.* $50 -$ *log.* $10.392 = 0.6822709$, therefore $p = \dfrac{0.6822709}{log.\,r}$; again, *log.* $r = 0.0170333$, therefore $p = \dfrac{0.06822709}{0.0170333} = 40.05$: and therefore the Advantage of paying half-yearly would amount to no more than gaining one Year in 41.

Quarterly Payments, or half-quarterly, nay even Payments made at every Inſtant of Time, would not much accelerate the Diſcharge of the Principal. Which to prove, let us reſume once more our general Theorem $\dfrac{A - \dfrac{A}{r^n}}{r-1} = P$; let us now imagine that the Number of Inſtants in the Year is $= t$, let us further ſuppoſe that s is the Number of Years in which the Principal will be diſcharged, then in the room of A, writing $\frac{1}{t}A$; in the room of r, writing $r^{\frac{1}{t}}$; and in the room of n, writing $s\,t$, we ſhall have $\dfrac{\frac{1}{t}A - \dfrac{\frac{1}{t}A}{r^s}}{r^{\frac{1}{t}} - 1} = P$. But it is known, that if t repreſents an infinite Number, ſuch as is the Number of Inſtants in one Year, then $r^{\frac{1}{t}} - 1 = \frac{1}{t}$ *log.* r, we have therefore $\dfrac{\frac{1}{t}A - \dfrac{\frac{1}{t}A}{r^s}}{\frac{1}{t}\,log.\,r} = P$, or $\dfrac{A - \dfrac{A}{r^s}}{log.\,r} = P$; let the Logarithm of r be ſuppoſed $= a$, therefore $A - \dfrac{A}{r^s} = a\,P$, and $\dfrac{A}{r^s} = A - a\,P$, and $r^s = \dfrac{A}{A - a\,P}$, which ſuppoſe $= \mathcal{Q}$, then $s = \dfrac{log.\,\mathcal{Q}}{log.\,r}$: But it is to be noted, that a repreſents the hyperbolic Logarithm of r, which is, as we have ſeen before, 0,0392207 when r ſtands for 1,04; this being ſuppoſed, the Logarithm of $\mathcal{Q}$ will be found to be 0,6663794, which being divided by the Logarithm of r *viz.* 0,0170333, the

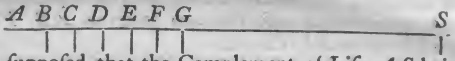
Quotient will be 39,1 Years; but in this laſt Operation the Logarithms of Q and r, may be taken out of a common Table.

CHAPTER VIII.

Containing the Demonſtration of what has been ſaid concerning the Probabilities of Survivorſhip.

What I call Complement of Life having been defined before *pag.* 265. I ſhall proceed to make uſe of that Word as often as occaſion ſhall require.

HYPOTHESIS.



Let it be ſuppoſed that the Complement of Life AS being divided into an infinite Number of equal Parts repreſenting Moments, the Probabilities of living from A to B, from A to C, from A to D, *&c.* are reſpectively proportional to the ſeveral Complements SB, SC, SD, in ſo much that theſe Probabilities may reſpectively be repreſented by the Fractions $\frac{SB}{SA}$, $\frac{SC}{SA}$, $\frac{SD}{SA}$, *&c.* This Hypotheſis being admitted the following Corollaries may be deduced from it.

COROLLARY I.

The Probability of Life's failing in any Interval of Time AF is meaſured by the Fraction $\frac{FA}{SA}$.

COROLLARY II.

When the Interval AF is once paſt, the Probability of Life's continuing from F to G is $\frac{SG}{SF}$, for at F, the Complement of Life is SF, and the Probability of its failing is $\frac{FG}{SF}$.

COROLLARY III.

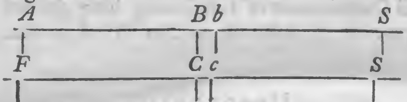
The Probability of Life's continuing from A to F, and then failing from F to G, is $\frac{SF}{SA} \times \frac{FG}{SF} = \frac{FG}{SA}$.

COROLLARY IV.

The Probability of Life's failing in any two or more equal Intervals of Time aſſigned between *A* and *S* are exactly the ſame, the Eſtimation being made at *A* conſidered as the preſent Time.

Theſe things premiſed, it will not be difficult to ſolve the following Problem.

Two Lives being given, to find the Probabilty of one of them fixed upon, ſurviving the other.



For, let the Complements of the two Lives be reſpectively $AS = n$ and $FS = p$, upon which take the two Intervals AB, $FC = z$, as alſo the two Moments Bb, $Cc = \dot{z}$.

The Probability of the firſt Life's continuing from *A* to *B*, or beyond it, is $\frac{n-z}{n}$; the Probability of the ſecond's continuing from *F* to *C*, and then failing in the Interval *Cc*, is by the third *Corollary* $\frac{\dot{z}}{p}$: therefore the Probability of the firſt Life's continuing during the time *AB* or beyond it, and of the ſecond's failing juſt at the end of that Time, is meaſured by $\frac{n-z}{n} \times \frac{\dot{z}}{p} = \frac{n\dot{z} - z\dot{z}}{np}$, whoſe Fluent $\frac{nz - \frac{1}{2}zz}{np}$ will expreſs the Probability of the firſt Life's continuing during any Interval of Time or beyond it, and of the ſecond's failing any time before or preciſely at the end of that Interval.

Let now p be written inſtead of z, and then the Probability of the firſt Life's ſurviving the ſecond, will be $\frac{np - \frac{1}{2}pp}{np} = 1 - \frac{\frac{1}{2}p}{n}$.

From the foregoing Concluſion we may immediately infer that the Probability of the ſecond Life's ſurviving the firſt is $\frac{\frac{1}{2}p}{n}$.

By the ſame method of arguing, we may proceed to the finding the Probability of any one of any Number of given Lives ſurviving all the reſt, and thereby verifying what we have ſaid in *Prob.* XVIII. and XIX.

CHAPTER IX.

Serving to render the Solutions in this Treatiſe more general, and more correct.

I.

Altho', in treating this ſubject of Annuities, I have made uſe only of Dr. *Halley*'s Table, founded upon the *Breſlaw* Bills of mortality; from which I deduced the *Hypotheſis* of an equable Decrement of Life: Yet are my Rules eaſily applicable to any other Table of Obſervations; by *Prob.* II. of my Letter to Mr. *Jones* in *Phil. Tranſ.* N°. 473, which the Reader may ſee below, in the *Appendix*.

Or inſtead of the Theorem there given, he may uſe that by which *Prob.* XXVI. was reſolved, which is rather more independent of *Tables:* And its application to our preſent purpoſe may be explained as follows.

As in all Tables of Obſervations deduced from Bills of mortality, or if we ſhould combine ſeveral of them into one, it will be found that, for certain Intervals at leaſt, *the Decrements of Life continue nearly the ſame*; if we conceive the whole Extent of Life to be repreſented by a right Line AZ, in which there are taken diſtances PQ, QR, RS, &c. proportional to thoſe Intervals, and at the points P, Q, R, S, &c. there be erected perpendiculars proportional to the Numbers of the Living at the beginning of the reſpective Intervals, and their Extremities are connected by right Lines; then there will be formed a *Polygon* Figure on the Baſe AZ, whoſe Ordinates will every where repreſent the Numbers of that *Table* from which the Figure was conſtructed; and the Inclinations of the Sides of the *Polygon* to its Baſe will expreſs the Convergencies of Life to its End, or the Degrees of Mortality belonging to the reſpective Intervals.

Say therefore, as the difference of the Ordinates at P and Q, is to the Ordinate at P: ſo is the Interval PQ, to a fourth PZ'; and PZ' ſhall be the *Complement* of Life at the age P; and the Point Z' in the Baſe ſhall be that from which the Complements are to be reckoned throughout the Interval PQ.

Let PZ', thus found, be ſubſtituted for n in the *Canon* of *Prob.* XXVI, and the Interval PQ for m, ſo ſhall the *Value* of that Interval be known: and in like manner the ſubſequent Values of QR, RS, &c. giving to each Interval its proper Complement QZ'', RZ''', &c.

And laſtly, theſe Values being ſeverally diſcounted, *Firſt*, in the Ratio of their reſpective Ordinates at *P*, *Q*, *R*, &c. to ſome preceding Ordinate as at *N*, at the Age 12, for inſtance; and *Secondly*, by the preſent Value of 1 *l* payable after the Years denoted by *NP*, *NQ*, *NR*, &c. their Sum will be the *Value* of the Life at *N*, according to the given Table of Obſervations. After which, the younger Lives muſt be computed from Year to Year: as thoſe after 70, or when an Interval contains but one Year, ought likewiſe to be computed.

If it is propoſed, for Example, to find how nearly my *Hypotheſis* agrees with Dr. *Halley*'s Table for the Interval of 8 Years between 33 and 41, it's Value, at 5 *per Cent.* computed by *Prob.* XXVI. will, to an Annuitant 33 Years old, be 5.9456, according to the *Hypotheſis*. But the Numbers of the Living at thoſe Ages being, in the Table, 507 and 436, if we compute immediately from it, we muſt take $n = \frac{507}{71} \times 8 = 57.14$; and the ſame Rule will give the Value 5.9831. Diſcount now the Values found as belonging to a Life of 12 Years; that is multiply the firſt by $\frac{53}{74}$, and the other by $\frac{507}{646}$; and the Products 4.2583 and 4.6957 diſcounted the *ſecond* time, that is, multiplied by .3589, the preſent Value of 1 *l.* payable after 21 (= 33 — 12) Years gives the Values 1.5283 and 1.6853; the difference being 0.157, near $\frac{1}{6}$ of a Year's purchaſe.

In general, the *Hypotheſis* will be found to give the Value of a *ſingle* Life, or of an aſſigned Interval, ſomewhat below what the *Table* makes it: but then, as both the young and the middle aged are obſerved to die off faſter in *England* than at *Breſlaw*, my Rules may very well be preferable, for the Purchaſes and Contracts that are made upon ſingle Lives in this Country.

In the ſame manner may any other *Tables* be compared with the *Hypotheſis*, and with one another. And if we give the preference to any particular Table, and would at the ſame time retain the *Hypotheſis* of equal Decrement we may, by the *differential Method*, eaſily find that *mean Termination* of Life, *Z*, which ſhall beſt correſpond to the *Table*.

II.

To preſerve ſomewhat of Elegance and Uniformity in my Solutions, as well as to avoid an inconvenient multiplicity of *Canons* and *Symbols*, I did transfer the Decrement of Life from an *Arithmetical* to a *Geometrical*

metrical Series: which however, in many Queſtions concerning *Combined* Lives, creates an error too conſiderable to be neglected. This hath not eſcaped the Obſervation of my Friends, no more than it had my own: but the ſame Perſons might have obſerved likewiſe, that ſuch Errors may, when it is thought neceſſary, be corrected by my own Rules; particularly upon this obvious principle, That, *if money is ſuppoſed to bear no Intereſt, the* Values *of Lives will coincide with what I call their* Expectations.

But as the Computation of ſuch Corrections might ſeem tedious; and becauſe practical Rules ought to be of ready Uſe, as well as ſufficiently exact; I chuſe rather to give another Rule for *joint Lives*, which will anſwer both theſe Purpoſes; at the ſame time that it is general, and eaſily retained in the Memory.

General Rule for the Valuation of joint Lives.

The given Ages being each increaſed by unity, find, by Problem XXI. *or* XXII. *the Number of Years due to their joint Continuance; and the Complement of twice this Number to* 86, *taken as a ſingle Life, will, in the proper Table, give nearly the Value required.*

EXAMPLE 1.

The Value of two joint Lives of 40 and 50, at 5 *per Cent.* was, in Prob. II. found to be 7.62. But if they are made 41 and 51, their joint *Expectation*, by *Prob.* XXI. will be 13 Years, theſe doubled and taken from 86 leave 60, againſt which in *Table* VIII, ſtands 8.39 Years purchaſe, nearly the Value ſought.

EXAMPLE 2.

The 3 joint Lives whoſe ſingle Values, at 4 *per Cent.* are 13, 14, 15 Years purchaſe, are in *Prob.* II. worth 7.41. But by *Table* VI, the Ages to which theſe Values belong, increaſed by Unity, are 42, 36, 28; whoſe Complements to 86 ſubſtituted for p, n, q, in $\frac{1}{2}p - \frac{p^2 \times \overline{n+q}}{6nq} + \frac{p^3}{12nq}$, the Canon for the *Expectation* of 3 joint Lives, gives 12.43. And 86 — 2 × 12.43 is nearly 61; at which Age a ſingle Life, in Table VI, is worth 8.75 Years purchaſe.

It is needleſs to add any thing concerning *longeſt Lives*, *Survivorſhips*, *Reverſions* and *Inſurances*; the Computation of their Values being only the combining thoſe of *ſingle* and *joint* Lives, by Addition and Subtraction: which being performed according to the Rules of this Treatiſe, the Anſwer may be depended upon as ſufficiently exact, in all uſeful Queſtions that can occur. For we do not here aim at an Accuracy beyond what the determination of our main *Data*, the Probabilities of human Life, and the conformity of our Hypotheſis to nature, can bear; nor do we give our Concluſions for perfectly exact, as is required in ſuch as are *purely* arithmetical, but only as very near Approximations; upon which buſineſs may be tranſacted, without conſiderable Loſs to any party concerned.

III.

The ſame Rule ſerves for the Caſe of an Annuity ſecured, upon *joint* Lives, by a Grant of Lands; or when the fractional part of the laſt Year is to be accounted for. Only, in this Caſe, 1°. *The Addition of Unity to each Life is to be omitted.* 2°. The ſingle Life is not now to be taken out of our Tables, or computed from $\frac{1-\frac{r}{n}P}{r-1}$ the Canon of *Prob.* I, but from $\frac{1}{r-1}-\frac{1}{an}P$, a being *Neper*'s Logarithm of r: as in *Phil. Tranſ.* N°. 473, and in *Chap.* I. foregoing.

According to which, if the Ages and Intereſt are as in Example 1; the *Expectation* of joint Life will be 13.3 Years; and thence $n = 26.6$; $P = 14.5358$; $a = .04879$: And the Value of the Annuity $20 - 11.2 = 8.8$; exceeding what it would have been upon yearly Payments by about $\frac{4}{10}$ of a Year's purchaſe.

And if the Payments are half yearly or quarterly, the ſkillful Computiſt cannot be at a loſs after what has been ſaid of thoſe Caſes in *Chap.* VII *.

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* See, on the Subject of Annuities, *Mathem. Repoſitory*, Vol. II. and III. by the ingenious Mr. *James Dodſon*, F. R. S.

F I N I S.

APPENDIX.

N°. I.

Dedication of the Firſt Edition of this Work (1718.)

TO

Sir ISAAC NEWTON, Kt. Preſident of the *Royal Society.*

SIR,

THE greateſt Help I have received in writing upon this Subject having been from your incomparable Works, eſpecially your Method of Series; I think it my Duty publickly to acknowledge, that the Improvements I have made, in the matter here treated of, are principally derived from yourſelf. The great benefit which has accrued to me in this reſpect, requires my ſhare in the general Tribute of Thanks due to you from the learned World: But one Advantage which is more particularly my own, is the Honour I have frequently had of being admitted to your private Converſation; wherein the Doubts I have had upon any Subject relating to *Mathematics*, have been reſolved by you with the greateſt Humanity and Condeſcenſion. Thoſe marks of your Favour are the more valuable to me, becauſe I had no other pretence to them but the earneſt deſire of underſtanding your ſublime and univerſally uſeful Speculations. I ſhould think my ſelf very happy, if having given my Readers a Method of calculating the Effects of Chance, as they are the reſult of Play, and thereby fixing certain Rules, for eſtimating how far ſome ſort of Events may rather be owing to Deſign than Chance, I could by this ſmall Eſſay excite in others a deſire of proſecuting theſe Studies, and of learning from your Philoſophy how to collect, by a juſt Calculation, the Evidences of exquiſite Wiſdom and Deſign, which appear in the *Phenomena* of Nature throughout the Univerſe. I am, with the utmoſt Reſpect,

Sir,

Your moſt humble,

and obedient Servant

U u A. de MOIVRE.

N°. II.

Note upon Coroll. 1. Prob. *VII*; *and upon* Prob. *IX*.

In that Corollary, it was found that the Probabilities of winning all each others Stakes being as $a^q \times \overline{a^p - b^p}$ and $b^p \times \overline{a^q - b^q}$; If we divide by $a-b$, and ſuppoſe the Chances for one Game to be equal, or $a = b$; then the Probabilities will be as the Number of pieces, or, in the Ratio of p to q.

Bnt when we have to divide ſuch Expreſſions continually, that is by ſome Power of $a-b$, as $\overline{a-b}|^2$, $\overline{a-b}|^4$, &c. it will be more convenient to uſe a *General Rule* for determining the Value of a Ratio whoſe Terms vaniſh by the contrariety of Signs. The Rule is this;

For the difference of the Quantities that deſtroy each other in any Caſe propoſed, write an indeterminate Quantity x; *in the Reſult reject all thoſe Terms that vaniſh when* x *becomes leſs than any finite Quantity: ſo ſhall the remaining homogeneous Terms, divided by their greateſt common Meaſure, expreſs the Ratio ſought.*

As in our example, if we make $a-b=x$, or $a=b+x$, and for a^p, a^q, write their equals $\overline{b+x}|^p$, $\overline{b+x}|^q$, expanded by the *Binomial* Theorem; the Ratio of *R* to *S*, in *Prob.* VII, will be reduced to that of $pb^{p+q-1} \times x + p . \frac{p-1}{2} + pq \times b^{p+q-2} \times x^2 +$ &c. to $qb^{p+q-1} \times x + q . \frac{q-1}{2} \times b^{p+q-2} \times x^2 +$ &c: Of which retaining only the two Terms that involve x, and dividing them by $b^{p+q-1} \times x$, we get $\frac{R}{S} = \frac{p}{q}$.

The Solution of *Prob.* IX. gives for the Gain of *A* the Product $\frac{qa^q \times \overline{a^p - b^p} - pb^p \times \overline{a^q - b^q}}{a^{p+q} - b^{p+q}}$ by $\frac{aG - bL}{a-b}$: and when $a=b$, if we ſubſtitute as before, the Terms involving x vaniſh in the Numerator of the firſt of theſe Factors; reducing it to $* \; * + \frac{pq}{2} \times \overline{p+q} \times b^{p+q-2} \times x^2 +$ &c: and the Denominator is $* \, \overline{p+q} \times b^{p+q-1} \times x +$ &c. The other Factor is $\frac{b \times \overline{G-L} + xG}{x}$, or when x vaniſhes with reſpect to b, $\frac{b \times \overline{G-L}}{x}$; and the Product of the two is $pq \times \frac{G-L}{2}$;

as in *Caſe* 2. *Caſe* 1 follows immediately from this; and the 3*d* has as little difficulty.

Another Example of our *Rule* may be; *To find, from the* Canon *of* Prob. *I. of the Treatiſe on* Annuities, *the* Expectation *of a Life whoſe Complement is* n; *that is, the preſent Value of a Rent or Annuity upon that Life, money bearing no Intereſt.* Now that *Canon* being $\frac{1-\frac{r}{n}P}{r-1}$, or $\frac{n-rP}{n\times\overline{r-1}}$, if for P we write its equal $\frac{1-r^{-n}}{r-1}$, and $1+x$ for r, the Value ſought will be $\frac{nr-n-r+r^{1-n}}{n\times\overline{r-1}|^{2}}=**+\frac{\overline{1-n}\times-\frac{n}{2}\times x^{2}+\&c.}{nx^{2}}=\frac{n-1}{2}$.

This Value wants half a year of $\frac{n}{2}$, its quantity according to the Rule given above, *pag.* 288: becauſe there the Probabilities of Life were ſuppoſed to decreaſe as the Ordinates of a Triangle; whereas, in the Hypotheſis of yearly payments in *Prob.* I, they decreaſe *per ſaltum*, like a Series of parallelograms inſcribed in a Triangle.

The Reader will likewiſe obſerve that our general Rule for computing the Value of a Fraction whoſe form becomes $\frac{0}{0}$, is in effect the ſame as that given by the Marquis *de l'Hoſpital* in his *Analyſe des infinimens petits.* And that, from the Number of Terms that vaniſh in the Operation, and from the *Sign* of the Term which determines the Ratio, the *Species* of algebraical *Curve* Lines, and the *Poſition* of their Branches, are diſcovered. See *Mac Laurin*'s Fluxions, Book I. Chap. 9. and Book II. Chap. 5.

N°. III.

Note to Prob. *XLV. from Mr.* Nicolas Bernoulli, Phil. Tranſ. 341.

To find the Probability that a *Poule* ſhall be ended in a given Number of Games: a Series of Fractions beginning with $\frac{1}{2^n - 1}$, whoſe Denominators increaſe in a double proportion, and the Numerator of each Fraction is the Sum of as many next preceding Numerators as there are Units in $n - 1$, will give the ſucceſſive Probabilities that the *Poule* ſhall be ended *preciſely* in n, $n+1$, $n+2$, $n+3$, &c. Games; and conſequently if as many Terms of this Series are added together, as there are units in $p+1$, their Sum will expreſs the Probability that the *Poule* ſhall be ended at leaſt in $n+p$ Games. For Example, if there are 4 Players, and thence $n=3$, we ſhall have this Series $\frac{1}{4}$, $\frac{1}{8}$, $\frac{2}{16}$, $\frac{3}{32}$, $\frac{5}{64}$, $\frac{8}{128}$, $\frac{13}{250}$, $\frac{21}{512}$, &c. Out of which if we form this other $\frac{1}{4}$, $\frac{3}{8}$, $\frac{8}{16}$, $\frac{19}{32}$, $\frac{43}{64}$, $\frac{94}{128}$, $\frac{201}{256}$, &c. whoſe Terms are the Sums of the Terms of former Series, theſe laſt will ſhew the Probability of the *Poule* ending in 3, 4, 5, 6, &c. Games, at leaſt.

N°. IV.

N°. IV.

A correct Table of the Sums of Logarithms, *from the Author's Supplement to his* Miſcellanea Analytica.

10	6.55976.30328.7678.	460	1026.82368.84245.7267.
20	18.38612.46168.7770.	470	1053.50280.26009.6230.
30	32.42366.00749.2572.	480	1080.27422.85779.2496.
40	47.91164.50681.5991.	490	1107.13604.49151.6763.
50	64.48307.48724.7209.	500	1134.08640.85351.3508.
60	81.92017.48493.9024.	510	1161.12355.00246.5923.
70	100.07840.50356.8004.	520	1188.24576.93048.6770.
80	118.85472.77224.9966.	530	1215.45143.16339.6251.
90	138.17193.57900.1086.	540	1242.73896.39114.8380.
100	157.97000.36547.1585.	550	1270.10685.12561.5931.
110	178.20091.76448.7008.	560	1297.55363.38324.8209.
120	198.82539.38472.1977.	570	1325.07790.39038.2121.
130	219.81069.31561.4815.	580	1352.67830.30922.0491.
140	241.12910.99886.9689.	590	1380 35351.98269.6983.
150	262.75689.34109.2616.	600	1408.10228.69662.7898.
160	284.67345.62406.8298.	610	1435.92337 95771.1124.
170	306.86078.19948.2847.	620	1463.81561.28607.3923.
180	329.30297.14247.9393.	630	1491.77784.02119.6951.
190	351.98588.98339.3535.	640	1519 80895.14015 3428.
200	374.89688.86400.4044.	650	1547.90787.08720.1888.
210	398.02458.26149.3624.	660	1576.07355.61385.9540.
220	421.35866.95421.3259.	670	1604.30499.62866.2770.
230	444.88978.26514 6048.	680	1632.60121.05589.2142.
240	468.60936.87056.4791.	690	1660 96124.70260.3147.
250	492.50958.63954.6190.	700	1689.38418.13336.1091.
260	516.58322.09826.1269.	710	1717.86911.55213 0134.
270	540.82361.20667.5295.	720	1746.41517.69081.2925.
280	565.22459.20470.1654.	730	1775.02151.70397.9157.
290	589.78043.33690.9860.	740	1803.68731.06935.9463.
300	614.48580.30437.7387.	750	1832.41175.49371.5144.
310	639.33572.32255.0106.	760	1861.19406.82372.5655.
320	664.32553.68741.5328.	770	1890.03348.96156 3791.
330	689.45087.77060.3828.	780	1918 92927.78485.4396.
340	714.70764.37846 5691.	790	1947.88071.07073.5663.
350	740.09197.42162.3279.	800	1976.88708.42376.3542.
360	765.60022 85067.1998.	810	2005.94771.20741.9152.
370	791.22896.82108 4658.	820	2035.06192 47899.6883.
380	816.97493.05636.3600.	830	2064.22906.92766.7182.
390	842.83506.38337.0506.	840	2093.44850 81552.2793.
400	868.80641.41777.2588.	850	2122.71961.92143.1027.
410	894.88621 38085.1630.	860	2152.04179 48752.7013.
420	921.07182.03166.5465.	870	2181.41444.16819.4477.
430	947.36071.70083.7526.	880	2210.83697.98139.1145.
440	973.75050.41416.4285.	890	2240.30884.26218.5633.
450	1000.23889.09583.9930.	900	2269.82947.61838.1577.

If we would examine theſe Numbers, or continue the Table farther on, we have that excellent Rule communicated to the Author by Mr. *James Stirling*; publiſhed in his Supplement to the *Miſcellanea Analytica*, and by Mr. *Stirling* himſelf in his *Methodus Differentialis*, Prop. XXVIII.

"Let $z - \frac{1}{2}$ be the laſt Term of any Series of the natural Num-
" bers $1, 2, 3, 4, 5 \ldots\ldots\ldots z - \frac{1}{2}$; $a = .43429448190325$ the reci-
" procal of *Neper*'s Logarithm of 10: Then three or four Terms of
" this Series $z \text{ Log. } z - az - \frac{a}{2.12z} + \frac{7a}{8.360z^3} - \frac{31a}{32.1260z^5} + \frac{127a}{128.1680z^7}$
" — &c. added to 0.399089934179, &c. which is half the Loga-
" rithm of a Circumference whoſe Radius is Unity, will be the Sum
" of the Logarithms of the given Series; or the Logarithm of the
" Product $1 \times 2 \times 3 \times 4 \times 5 \text{ - - - - - } \times \overline{z - \frac{1}{2}}$."

The Coefficients of all the Terms after the firſt two being formed as follows.

$$\text{Put } -\frac{1}{3.4} = A$$

$$-\frac{1}{5.8} = A + 3B$$

$$-\frac{1}{7.12} = A + 10B + 5C$$

$$-\frac{1}{9.16} = A + 21B + 35C + 7D$$

$$-\frac{1}{11.20} = A + 36B + 126C + 84D + 9E.$$

&c.

In which the Numbers 1, 1, 1, &c. 3, 10, 21, 36, &c. 5, 35, 126, &c. that multiply *A*, *B*, *C*, &c. are the alternate *Unciæ* of the odd Powers of a Binomial. Then the Coefficients of the ſeveral Terms will be $\frac{1}{2} \times A = -\frac{1}{2.12}$, $\overline{\frac{1}{2}}\Big|^3 \times B = \frac{7}{8.360}$, $\overline{\frac{1}{2}}\Big|^5 \times C = \frac{31}{32.1260}$, &c. See the general Theorem and Demonſtration in Mr. *Stirling*'s Propoſition quoted above.

N°. V.

Some Uſeful Cautions.

One of the moſt frequent occaſions of Error in managing Problems of Chance, being to allow more or fewer Chances than really there are; but more eſpecially in the firſt Caſe, for the fault lies commonly that way, I have in the Introduction taken great care to ſettle the Rules of proceeding cautiouſly in this matter; however it will not be amiſs to point out more particularly the danger of being miſtaken.

Suppoſe therefore I have this Queſtion propoſed; There are two Parcels of three Cards, the firſt containing King, Queen, and Knave of Hearts, the ſecond the King, Queen, and Knave of Diamonds, and that I were promiſed the Sum S, in caſe that in taking a Card out of each Parcel, I ſhould take out either the King of Hearts, or the King of Diamonds, and that it were required I ſhould determine the value of my Expectation.

If I reaſon in this manner; the Probability of taking out the King of Hearts is $\frac{1}{3}$, therefore $\frac{1}{3}s$ is my due upon that acccunt; the Probability of taking out the King of Diamonds is alſo $\frac{1}{3}$, and therefore that part of my Expectation is $\frac{1}{3}s$ as the other was, and conſequently my whole Expectation is $\frac{2}{3}s$; this would not be a legitimate way of reaſoning: for I was not promiſed that in caſe I ſhould take out both Kings, I ſhould have the Sum $2s$, but barely the Sum s. Therefore we muſt argue thus; the Probability of taking out the King of Hearts is $\frac{1}{3}$, the probability of miſſing the King of Diamonds is $\frac{2}{3}$, and therefore the probability of taking out the King of Hearts, and miſſing the King of Diamonds is $\frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$, for which reaſon that part of my Expectation which ariſes from the probability of taking out the King of Hearts, and miſſing the King of Diamonds is $\frac{2}{9}s$; for the ſame reaſon that part of my Expectation which ariſes from the probability of taking the King of Diamonds and miſſing the King of Hearts is $\frac{2}{9}s$, but I ought not to be deprived of the Chance of taking out the two Kings of which the probability is $\frac{1}{9}$, and therefore the value of that Chance is $\frac{1}{9}s$; for which reaſon, the value of my whole Expectation is $\frac{2}{9}s + \frac{2}{9}s + \frac{1}{9}s = \frac{5}{9}s$ which is leſs by $\frac{1}{9}s$ than $\frac{2}{3}s$.

But ſuppoſe I were propoſed to have $2s$ given me in caſe I took out both Kings, then this laſt Expectation would be $\frac{2}{9}s$, which would make the whole value of my Expectation to be $\frac{2}{9}s + \frac{2}{9}s + \frac{2}{9}s = \frac{6}{9}s = \frac{2}{3}s$.

One may perceive by this ſingle inſtance, that when two Events are ſuch, that on the happening of either of them I am to have a Sum $ſ$, the probability of that Chance ought to be eſtimated by the Sum of the Probabilities of the happening of each, wanting the probability of their both happening.

But not to argue from particulars to generals. Let x be the probability of the happening of the firſt, and y the probability of the happening of the ſecond, then $x \times \overline{1 - y}$ or $x - xy$ will repreſent the probability of the happening of the firſt and failing of the ſecond, and $y \times \overline{1 - x}$ or $y - xy$ will repreſent the probability of the happening of the ſecond and failing of the firſt, but xy repreſents the happening of both; and therefore $x - xy + y - xy + xy$ or $x + y - xy$ will repreſent the probability of the happening of either.

This concluſion may be confirmed thus; $1 - x$ being the probability of the firſt's failing, and $1 - y$ the probability of the ſecond's failing, then the Product $\overline{1 - x} \times \overline{1 - y}$ or $1 - x - y + xy$ will repreſent the probability of their both failing; and this being ſubtracted from Unity, the remainder, *viz.* $x + y - xy$ will repreſent the probability of their not both failing, that is of the happening of either.

And if there be three Events concerned, of which the Probabilities of happening are reſpectively x, y, z, then multiplying $1 - x$ by $1 - y$ and that again by $1 - z$, and ſubtracting the Product from Unity, the remainder will expreſs the probability of the happening of one at leaſt of them, which conſequently will be $x + y + z - xy - xz - yz + xyz$; and this may be purſued as far as one pleaſes.

A difficulty almoſt of the ſame nature as that which I have explained is contained in the two following Queſtions: the firſt is this;

A Man throwing a Die ſix times is promiſed the Sum $ſ$ every time he throws the Ace, to find the value of his Expectation.

The ſecond is this; a Man is promiſed the Sum $ſ$ if at any time in ſix trials he throws the Ace, to find the value of his Expectation.

In the firſt Queſtion every throw independently from any other is entitled to an Expectation of the Sum $ſ$, which makes the value of the Expectation to be $\frac{1}{6}ſ + \frac{1}{6}ſ + \frac{1}{6}ſ + \frac{1}{6}ſ + \frac{1}{6}ſ + \frac{1}{6}ſ = ſ$; but in the ſecond, none but the firſt throw is independent, for the ſecond has no right but in caſe the firſt has failed, nor has the third any right but in caſe the two firſt have failed, and ſo on; and therefore the value of the Expectation being the Sum expected, multiplied by the Sum of the Probabilities of the Ace's being thrown at any time, excluſive of the Probabilities of its having been thrown before, will be $\frac{1}{6}\text{ſ} + \frac{5}{36}\text{ſ} + \frac{25}{216}\text{ſ} + \frac{125}{1296}\text{ſ} + \frac{625}{7776}\text{ſ} + \frac{3125}{46656}\text{ſ} = \frac{31031}{46656}$ that is nearly $\frac{2}{3}\text{ſ}$.

We may alſo proceed thus; the probability of the Ace's being miſſed ſix times together is $\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{15625}{46656}$, and therefore the probability of its not being miſſed ſix times, that is of its happening ſome time or other in 6 throws is $1 - \frac{15625}{46656} = \frac{31031}{46656}$, and conſequently the value of the Expectation is $\frac{31031}{46656}$ as it was found before.

Another Inſtance may be, the computing the Odds of the Bet, *That one of the* 4 *Players at* Whiſt *ſhall have above* 4 *Trumps*. The Solution one might think was by adding all the Chances (in the Tables pag. 177) which the 4 Gameſters have for 5 or more Trumps; and this would be true, were every Gameſter to lay for himſelf in particular. But as it may happen that *two* of the Gameſters have above 4 Trumps, and yet, as the Bet is commonly laid, only one Stake is paid, half the Number of theſe laſt Chances (computed by *Prob.* XX.) is to be ſubtracted: which reduces the Wager nearly to an equality.

N°. VI.

A ſhort method of calculating the value of Annuities on Lives, from Tables of Obſervations, In a Letter to W. Jones *Eſq*; Phil. Tranſ. *N*°. 473.

Although it has been an eſtabliſhed cuſtom, in the payment of Annuities on Lives, that the laſt rent is loſt to the heirs of the late poſſeſſor of an annuity, if the perſon happens to die before the expiration of the term agreed on for payment, whether yearly, half-yearly, or quarterly: nevertheleſs, in this Paper I have ſuppoſed, that ſuch a part of the rent ſhould be paid to the heirs of the late poſſeſſor, as may be exactly proportioned to the time elapſed between that of the laſt payment, and the very moment of the Life's expiring; and this by a proper, accurate, and geometrical calculation.

I have been induced to take this method, for the following reaſons; firſt, by this ſuppoſition, the value of Lives would receive but an inconſiderable increaſe; ſecondly, by this means, the ſeveral intervals of life, which, in the Tables of Obſervations, are found to have uniform decrements, may be the better connected together. It is with this view that I have framed the two following Problems, with their Solutions.

PROBLEM I.

To find the value of an Annuity, ſo circumſtantiated, that it ſhall be on a Life of a given age; and that upon the failing of that life, ſuch a part of the rent ſhall be paid to the heirs of the late poſſeſſor of an Annuity, as may be exactly proportioned to the time intercepted between that of the laſt payment, and the very moment of the life's failing.

SOLUTION.

Let n repreſent the complement of life, that is, the interval of time between the given age, and the extremity of old-age, ſuppoſed at 86.

r the amount of 1 *l.* for one year.

α the Logarithm of r.

P the preſent value of an Annuity of 1 *l.* for the given time.

Q the value of the life ſought.

Then $\frac{1}{r-1} - \frac{P}{\alpha n} = Q$.

DEMONSTRATION.

For, let z repreſent any indeterminate portion of n. Now the Probability of the life's attaining the end of the interval z, and then failing, is to be expreſſed by $\frac{z}{n}$, (as ſhewn in my book of Annuities upon Lives) upon the ſuppoſition of a perpetual and uniform decrement of life.

But it is well known, that if an Annuity certain of 1 *l.* be paid during the time z, its preſent value will be $P = \frac{1-r^{\frac{1}{z}}}{r-1}$ or $\frac{1}{r-1} - \frac{1}{\overline{r-1}\times r^{z}}$.

And, by the laws of the Doctrine of Chances, the Expectation of ſuch a life, upon the preciſe interval $\dot{z}$, will be expreſſed by $\frac{\dot{z}}{n\times\overline{r-1}} - \frac{\dot{z}}{nr^{z}\times\overline{r-1}}$; which may be taken for the ordinate of a curve, whoſe area is as the value of the life required.

In order to find the area of this curve, let $p = n \times \overline{r - 1}$; and then the ordinate will become $\frac{\dot{z}}{p} - \frac{\dot{z}}{pr^z}$, a much more commodious expreſſion.

Now it is plain, that the fluent of the firſt part is $\frac{z}{p}$: but as the fluent of the ſecond part is not ſo readily diſcovered, it will not be improper, in this place, to ſhew by what artifice I found it; for I do not know, whether the ſame method has been made uſe of by others: all that I can ſay, is, that I never had occaſion for it, but in the particular circumſtance of this Problem.

Let, therefore, $r^z = x$; hence z Log. $r =$ Log. x; therefore $\dot{z}$ Log. $r =$ (Fluxion of the Log. $x =$) $\frac{\dot{x}}{x}$, or $\alpha \dot{z} = \frac{\dot{x}}{x}$; conſequently $\dot{z} = \frac{\dot{x}}{\alpha x}$, and $\frac{\dot{z}}{r^z} = \frac{\dot{x}}{\alpha xx}$: but the fluent of $\frac{\dot{x}}{\alpha xx}$ is $(-\frac{1}{\alpha x} =) - \frac{1}{\alpha r^z}$; and therefore the fluent of $- \frac{\dot{z}}{pr^z}$ will be $+ \frac{1}{p\alpha r^z}$.

The ſum of the two fluents will be $\frac{z}{p} + \frac{1}{p\alpha r^z}$; but, when $z = 0$, the whole fluent ſhould be $= 0$; let therefore the whole fluent be $\frac{z}{p} + \frac{1}{p\alpha r^z} + q = 0$.

Now, when $z = 0$, then $\frac{z}{p} = 0$, and $\frac{1}{\alpha pr^z}$ becomes $\frac{1}{\alpha p}$ (for $r^z = 1$,) conſequently $\frac{1}{\alpha p} + q = 0$; and $q = -\frac{1}{\alpha p}$: therefore the area of a curve, whoſe ordinate is $\frac{\dot{z}}{p} - \frac{\dot{z}}{pr^z}$ will be $(\frac{z}{p} - \frac{1}{\alpha p} + \frac{1}{\alpha pr^z} =) \frac{z}{p} - \overline{1 - \frac{1}{r^z}} \times \frac{1}{\alpha p}$.

But $P = \frac{1}{r-1} - \frac{1}{\overline{r-1} \times r^z}$; therefore $1 - \frac{1}{r^z} = \overline{r - 1} \times P$, and the expreſſion for the area becomes $\frac{z}{n \times \overline{r-1}} - \frac{P}{\alpha n}$: And putting n inſtead of z, that area, or the value of the life, will be expreſſed by $\frac{1}{r-1} - \frac{P}{\alpha n}$. *Q. E. D.*

Thoſe who are well verſed in the nature of Logarithms, I mean thoſe that can deduce them from the Doctrine of Fluxions and infinite Series, will eaſily apprehend, that the quantity here called α, is that which ſome call the hyperbolic Logarithm; others, the natural Logarithm: it is what Mr. *Cotes* calls the Logarithm whoſe modulus is 1: laſtly, it is by ſome called *Neper*'s Logarithm. And, to ſave the reader ſome trouble in the practice of this laſt theorem, the moſt neceſſary natural Logarithms, to be made uſe of in the preſent diſquiſition about Lives, are the following:

If $r = 1.04$, then will $\alpha = 0.0392207$.
$r = 1.05$, - - - $\alpha = 0.0487901$.
$r = 1.06$, - - - $\alpha = 0.0582589$.

It is to be obſerved, that the Theorem here found makes the Values of Lives a little bigger, than what the Theorem found in the firſt Problem of my book of Annuities on Lives, does; for, in the preſent caſe, there is one payment more to be made, than in the other; however, the difference is very inconſiderable.

But, although it be indifferent which of them is uſed, on the ſuppoſition of an equal decrement of life to the extremity of old-age; yet, if it ever happens, that we ſhould have Tables of Obſervations, concerning the mortality of mankind, intirely to be depended upon, then it would be convenient to divide the whole interval of life into ſuch ſmaller intervals, as, during which, the decrements of life have been obſerved to be uniform, notwithſtanding the decrements in ſome of thoſe intervals ſhould be quicker, or ſlower, than others; for then the Theorem here found would be preferable to the other; as will be ſhewn hereafter.

That there are ſuch intervals, Dr. *Halley*'s Tables of Obſervations ſufficiently ſhew; for inſtance; out of 302 perſons of 54 years of age, there remain, after 16 years (that is, of the age of 70) but 142; the decrements from year to year having been conſtantly 10; and the ſame thing happens in other intervals; and it is to be preſumed, that the like would happen in any other good Tables of Obſervations.

But, in order to ſhew, in ſome meaſure, the uſe of the preceding Theorem, it is neceſſary to add another Problem; which, though its Solution is to be met with in the firſt edition of my book of Annuities on Lives, yet it is convenient to have it inſerted here, on account of the connexion that the application of the preceding Problem has with it.

In the mean time, it will be proper to know, *What part of the yearly rent ſhould be paid to the heirs of the late poſſeſſor of an Annuity, as may be exactly proportioned to the time elapſed between that of the laſt payment, and the very moment of the life's expiring.* To determine this, put A for the yearly rent; $\frac{1}{m}$ for the part of the year intercepted between the time of the laſt payment, and the inſtant of the life's failing; r the amount of 1 *l.* at the year's end: then will $\frac{r^{\frac{1}{m}} - 1}{r - 1} A$ be the ſum to be paid.

PRO-

PROBLEM II.

To find the Value of an Annuity for a limited interval of life, during which the decrements of life may be conſidered as equal.

SOLUTION.

Let a and b repreſent the number of people living in the beginning and end of the given interval of years.

s repreſent that interval.

P the Value of an Annuity certain for that interval.

Q the Value of an Annuity for life ſuppoſed to be neceſſarily extinct in the time s; or (which is the ſame thing) the Value of an Annuity for a life, of which the complement is s.

Then $Q + \frac{b}{a} \times \overline{P - Q}$ will expreſs the Value required.

DEMONSTRATION.

For, let the whole interval between a and b be filled up with arithmetical mean proportionals; therefore the number of people living in the beginning and end of each year of the given interval s will be repreſented by the following Series; *viz.*

$$a . \frac{sa - a + b}{s} . \frac{sa - 2a + 2b}{s} . \frac{sa - 3a + 3b}{s} . \frac{sa - 4a + 4b}{s} . \text{ \&c. to } b.$$

Conſequently, the Probabilities of the life's continuing during 1, 2, 3, 4, 5, *&c.* years will be expreſſed by the Series,

$$\frac{sa - a + b}{sa} . \frac{sa - 2a + 2b}{sa} . \frac{sa - 3a + 3b}{sa} . \frac{sa - 4a + 4b}{sa} . \text{ \&c. to } \frac{b}{a}.$$

Wherefore, the Value of an Annuity of 1 *l.* granted for the time s, will be expreſſed by the Series

$$\frac{sa - a + b}{sar} + \frac{sa - 2a + 2b}{sar^2} + \frac{sa - 3a + 3b}{sar^3} + \frac{sa - 4a + 4b}{sar^4}, \text{ \&c. to } + \frac{b}{ar^s};$$

this Series is diviſible into two other Series's, *viz.*

$$1\text{ſt.}\ \frac{s-1}{sr} + \frac{s-2}{sr^2} + \frac{s-3}{sr^3} + \frac{s-4}{sr^4}, \text{ \&c. to } + \frac{s-s}{sr^s}.$$

$$2\text{d.}\ \frac{b}{a} \times \overline{\frac{1}{sr} + \frac{2}{sr^2} + \frac{3}{sr^3} + \frac{4}{sr^4}, \text{ \&c. to } \frac{s}{sr^s}}.$$

Now, ſince the firſt of theſe Series's begins with a Term whoſe Numerator is $s-1$, and the ſubſequent Numerators each decreaſe by unity; it follows, that the laſt Term will be $= 0$; and conſequently, that Series expreſſes the Value of a life neceſſarily to be extinct in the time s. The ſum of which Series may be eſteemed as a given quantity; and is what I have expreſſed by the ſymbol Q in Problem 1.

The ſecond Series is the difference between the two following Series's,

$$\frac{b}{a} \times \overline{\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \frac{1}{r^4} + \text{\&c. to } \frac{1}{r^s}}.$$

$$\frac{b}{a} \times \overline{\frac{s-1}{sr} + \frac{s-2}{sr^2} + \frac{s-3}{sr^3} + \frac{s-4}{sr^4} \text{\&c. to} + \frac{s-s}{sr^s}}.$$

Where, neglecting the common multiplier $\frac{b}{a}$, the firſt Series is the Value of an Annuity certain to continue s years; which every mathematician knows how to calculate, or is had from Tables already compoſed for that purpoſe: this Value is what I have called P; and the ſecond Series is Q.

Therefore $Q + \frac{b}{a} \times \overline{P - Q}$ will be the Value of an Annuity on a life for the limited time. *Q. E. D.*

It is obvious, that the Series denoted by Q, muſt of neceſſity have one Term leſs than is the number of equal intervals contained in s; and therefore, if the whole extent of life, beginning from an age given, be divided into ſeveral intervals, each having its own particular uniform decrements, there will be, in each of theſe intervals, the defect of one payment; which to remedy, the Series Q muſt be calculated by Problem 1.

EXAMPLE.

To find the Value of an Annuity for an age of 54, *to continue* 16 *years, and no longer.*

It is found, in Dr. *Halley*'s Tables of Obſervations, that a is 302, and b 172: now $n = s = 16$; and, by the Tables of the Values of Annuities certain, $P = 10.8377$; alſo (by Problem 1.) $Q = (\frac{1}{r-1} - \frac{P}{an} =)$ 6.1168. Hence it follows (by this Problem), that the Value of an Annuity for an age of 54, to continue during the limited time of 16 years, ſuppoſing intereſt at 5 *per cent. per annum*, will be worth $(Q + \frac{b}{a} \times \overline{P - Q} =)$ 8.3365 years purchaſe.

From Dr. *Halley*'s Tables of Obſervations, we find, that from the age of 49 to 54 incluſive, the number of perſons, exiſting at thoſe ſeveral ages, are, 357, 346, 335, 324, 313, 302, which comprehends a ſpace of five years; and, following the precepts before laid down, we ſhall find, that an Annuity for a life of 49, to continue for the limited time of 5 years, intereſt being at 5 *per cent. per annum*, is worth 4.0374 years purchaſe.

And, in the ſame manner, we ſhall find, that the Value of an Annuity on a life, for the limited time comprehended between the ages of 42 and 49, is worth 5.3492 years purchaſe.

Now, if it were required to determine the Value of an Annuity on life, to continue from the age of 42 to 70, we muſt proceed thus:

It has been proved, that an Annuity on life, reaching from the age of 54 to 70, is worth 8.3365 years purchaſe; but this Value, being eſtimated from the age of 49, ought to be diminiſhed on two accounts: Firſt, becauſe of the Probability of the life's reaching from 49 to 54, which Probability is to be deduced from the Table of Obſervations, and is proportional to the number of people living at the end and beginning of that interval, which, in this caſe, will be found 302 and 357: The ſecond diminution proceeds from a diſcount that ought to be made, becauſe the Annuity, which reaches from 54 to 70, is eſtimated 5 years ſooner, *viz.* from the age of 49, and therefore that diminution ought to be expreſſed by $\frac{1}{r^5}$; ſo that the total diminution of the Annuity of 16 years will be expreſſed by the fraction $\frac{302}{357r^5}$, which will reduce it from 8.3365 years purchaſe to 5.5259; this being added to the Value of the Annuity to continue from 49 to 54, *viz.* 4.0374, will give 9.5633, the Value of an Annuity to continue from the age of 49 to 70. For the ſame reaſon, the Value 9.5633, eſtimated from the age of 42, ought to be reduced, both upon account of the Probability of living from 42 to 49, and of the diſcount of money for 7 years, at 5 *per cent. per annum*, amounting together to 3.8554, which will bring it down to 5.7079; to this adding the Value of an Annuity on a life to continue from the age of 42 to 49, found before to be 5.3492, the ſum will be 11.0571 years purchaſe, the Value of an Annuity to continue from the age of 42 to 70.

In the ſame manner, for the laſt 16 years of life, reaching from 70 to 86, when properly diſcounted, and alſo diminiſhed upon the account of the Probability of living from 42 to 70, the Value of thoſe laſt 16 years will be reduced to 0.8; this being added to 11.0571 (the Value of an Annuity to continue from the age of 42 to 70, found before), the ſum will be 11.8571 years purchaſe, the Value of an Annuity to continue from the age of 42 to 86; that is, the Value of an Annuity on a life of 42; which, in my Tables, is but 11.57, upon the ſuppoſition of an uniform decrement of life, from an age given to the extremity of old-age, ſuppoſed at 86.

It is to be obſerved, that the two diminutions, above-mentioned, are conformable to what I have ſaid in the Corollary to the ſecond Problem of the firſt edition, printed in the year 1724.

Thoſe who have ſufficient leiſure and ſkill to calculate the Value of joint Lives, whether taken two and two, or three and three, in the ſame manner as I have done the firſt Problem of this tract, will be greatly aſſiſted by means of the two following Theorems:

If the ordinate of a curve be $\frac{z}{r^z}$; its area will be $\frac{1}{a^2} - \frac{1}{a^2 r^z} - \frac{z}{ar^z}$.

If the ordinate of a curve be $\frac{z^2}{r^z}$; its area will be $\frac{2}{a^3} - \frac{2}{a^3 r^z} - \frac{2z}{a^2 r^z} - \frac{z^2}{ar^z}$.

N°. VII.

N°. VII.

The Probabilities of human Life, according to different Authors.

Table I, by Dr. *Halley.*

Age	Living.	Age	Living.	Age	Living.	Age	Living.	Age	Living.	Age	Living.
1	1000	16	622	31	523	46	387	61	232	76	78
2	855	17	616	32	515	47	377	62	222	77	68
3	798	18	610	33	507	48	367	63	212	78	58
4	760	19	604	34	499	49	357	64	202	79	* 49
5	732	20	598	35	*490	50	*346	65	192	80	41
6	710	21	592	36	481	51	335	66	182	81	34
7	692	22	586	37	472	52	324	67	172	82	28
8	680	23	580	38	463	53	313	68	162	83	23
9	670	24	574	39	454	54	302	69	152	84	19
10	661	25	*567	40	445	55	*292	70	142	*	*
11	653	26	560	41	436	56	282	71	*131		
12	646	27	553	42	427	57	272	72	120		
13	*640	28	546	43	*417	58	262	73	109		
14	634	29	539	44	407	59	252	74	98		
15	628	30	*531	45	397	60	242	75	* 88		

Table II. by M. *Kerſſeboom.*

Age	Living.	Age	Living.	Age	Living.	Age	Living.	Age	Living.	Age	Living.	Age	Living.
0	1400												
1	1125	16	849	31	699	46	550	61	369	76	160	91	7
2	1075	17	842	32	687	47	540	62	356	77	145	92	5
3	1030	18	835	33	675	48	530	63	343	78	130	93	3
4	993	19	826	34	665	49	518	64	329	79	115	94	2
5	964	20	817	35	655	50	507	65	315	80	100	95	1
6	947	21	808	36	645	51	495	66	301	81	87	96	0.6
7	930	22	800	37	635	52	482	67	287	82	75	97	0.5
8	913	23	792	38	625	53	470	68	273	83	64	98	0.4
9	904	24	783	39	615	54	458	99	259	84	55	99	0.2
10	895	25	772	40	605	55	446	70	245	85	45	100	0.0
11	886	26	760	41	596	56	434	71	231	86	36	*	
12	878	27	747	42	587	57	421	72	217	87	28		
13	870	28	735	43	578	58	408	73	203	88	21		
14	863	29	723	44	569	59	395	74	189	89	15		
15	856	30	711	45	560	60	382	75	175	90	10		

Table III. by M. de *Parcieux.*

Age	Living	Age	Living.	Age	Living.	Age	Living.	Age	Living.
1	****	21	806	41	650	61	450	81	101
2	****	22	798	42	643	62	437	82	85
3	1000	23	790	43	636	63	423	83	71
4	970	24	782	44	629	64	409	84	59
5	948	25	774	45	622	65	395	85	48
6	930	26	766	46	615	66	380	86	38
7	915	27	758	47	607	67	364	87	29
8	902	28	750	48	599	68	347	88	22
9	890	29	742	49	590	69	329	89	16
10	880	30	734	50	581	70	310	90	11
11	872	31	726	51	571	71	291	91	7
12	866	32	718	52	560	72	271	92	4
13	860	33	710	53	549	73	251	93	2
14	854	34	702	54	538	74	231	94	1
15	848	35	694	55	526	75	211	95	0
16	842	36	686	56	514	76	192	96	*
17	835	37	678	57	502	77	173	97	*
18	828	38	671	58	489	78	154	98	
19	821	39	664	59	476	79	136	99	
20	814	40	657	60	463	80	118	100	

Table IV. by Meſſieurs *Smart* and *Simpſon.*

Age	Living.	Age	Living.	Age	Living.	Age	Living.	Age	Living.
1	1280 870	17	480	33	358	49	212	65	99
2	700	18	474	34	349	50	204	66	93
3	635	19	468	35	340	51	196	67	87
4	600	20	462	36	331	52	188	68	81
5	580	21	455	37	322	53	180	69	75
6	564	22	448	38	313	54	172	70	69
7	551	23	441	39	304	55	165	71	64
8	541	24	434	40	294	56	158	72	59
9	532	25	426	41	284	57	151	73	54
10	524	26	418	42	274	58	144	74	49
11	517	27	410	43	264	59	137	75	45
12	510	28	402	44	255	60	130	76	41
13	504	29	394	45	246	61	12	77	38
14	498	30	385	46	237	62	117	78	35
15	492	31	376	47	228	63	111	79	32
16	486	32	367	48	220	64	105	80	29

Remarks

Remarks on the foregoing Tables.

The firſt Table is that of Dr. *Halley*, compoſed from the Bills of Mortality of the City of *Breſlaw*; the beſt, perhaps, as well as the firſt of its kind; and which will always do honour to the judgment and ſagacity of its excellent Author.

Next follows a Table of the ingenious Mr. *Kerſſeboom*, founded chiefly upon Regiſters of the *Dutch* Annuitants, carefully examined and compared, for more than a century backward. And *Monſieur* de *Parcieux* by a like uſe of the Liſts of the *French Tontines*, or *long Annuities*, has furniſhed us Table III; whoſe numbers were likewiſe verified upon the *Necrologies* or mortuary Regiſters of ſeveral religious houſes of both Sexes.

To theſe is added the Table of Meſſieurs *Smart* and *Simpſon*, adapted particularly to the City of *London*; whoſe inhabitants, for reaſons too well known, are ſhorter lived than the reſt of mankind.

Each of theſe Tables may have its particular uſe: The *Second* or *Third* in valuing the *better* ſort of Lives, upon which one would chuſe to hold an Annuity; the *Fourth* may ſerve for *London*, or for Lives ſuch as thoſe of its Inhabitants are ſuppoſed to be: while Dr *Halley*'s numbers, falling between the two Extremes, ſeem to approach nearer to the general courſe of nature. And in Caſes of *combined* Lives, two or more of the Tables may perhaps be uſefully employed.

Beſides theſe, the celebrated Monſieur *de Buffon* † has lately given us a new Table, from the actual Obſervations of Monſieur *du Pré de S. Maur* of the *French Academy*. This Gentleman, in order to ſtrike a juſt *mean*, takes three populous pariſhes in the City of *Paris*, and ſo many country Villages as furniſh him nearly an equal number of Lives: and his care and accuracy in that performance have been ſuch as to merit the high approbation of the learned Editor. It was therefore propoſed to add this Table to the reſt; after having purged its numbers of the inequalities that neceſſarily happen in fortuitous things, as well as of thoſe ariſing from the careleſs manner in which *Ages* are given in to the pariſh Clerks; by which the years that are multiples of 10 are generally overloaded.

But this having been done with all due care, and the whole reduced to Dr. *Halley*'s Denomination of 1000 Infants of a year old; there reſulted only a mutual confirmation of the two Tables; Mr. *du Pré*'s Table making the Lives ſomewhat better as far as 39 years, and thence a ſmall matter worſe than they are by Dr. *Halley*'s.

We may therefore retain this laſt as no bad ſtandard for mankind in *general*; till a better *Police*, in this and other nations, ſhall furniſh the proper *Data* for correcting it, and for expreſſing the Decrements of Life more accurately, and in larger numbers.

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† *Hiſtoire Naturelle*, Tome II.

For which purpoſe, the pariſh Regiſters ought to be kept in a better manner, according to one or other of the Forms that have been propoſed by Authors. Or, if we ſuppoſe the numbers annually born to have been nearly the ſame for an age paſt, the thing may be done at once, by taking the numbers of the living, with their ages, throughout every Pariſh in the Kingdom: as was in part ordered ſome time ago by the Right Reverend the Biſhops: but their Order was not univerſally obeyed; for what reaſon we pretend not to gueſs. Certain it is, that a *Cenſus* of this kind once eſtabliſhed, and repeated at proper intervals, would furniſh to our Governours, and to ourſelves, much important inſtruction of which we are now in a great meaſure deſtitute: Eſpecially if the whole was diſtributed into the proper *Claſſes* of *married* and *unmarried*, *induſtrious* and *chargeable* Poor, *Artificers* of every kind, *Manufacturers*, &c. and if this was done in each County, City, and Borough, ſeparately; that particular uſeful concluſions might thence be readily deduced; as well as the general ſtate of the Nation diſcovered; and the Rate according to which *human Life* is waſting from year to year. See, on this ſubject, the judicious Obſervations of Mr. *Corbyn Morris*, addreſſed to *Thomas Potter* Eſq; in the year 1751.

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